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2010 J. Phys. A: Math. Theor. 43 105101

(http://iopscience.iop.org/1751-8121/43/10/105101)

J. Phys. A: Math. Theor. **43** (2010) 105101 (25pp) doi:10.1088/1751-8113/43/10/105101

Synchronization of Markovian jumping stochastic complex networks with distributed time delays and probabilistic interval discrete time-varying delays

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Received 9 October 2009, in final form 10 January 2010
Published 18 February 2010
Online at stacks.iop.org/JPhysA/43/105101

Abstract

The paper investigates the synchronization stability problem for a class of complex dynamical networks with Markovian jumping parameters and mixed time delays. The complex networks consist of m modes and the networks switch from one mode to another according to a Markovian chain with known transition probability. The mixed time delays are composed of discrete and distributed delays, the discrete time delay is assumed to be random and its probability distribution is known *a priori*. In terms of the probability distribution of the delays, the new type of system model with probability-distribution-dependent parameter matrices is proposed. Based on the stochastic analysis techniques and the properties of the Kronecker product, delay-dependent synchronization stability criteria in the mean square are derived in the form of linear matrix inequalities which can be readily solved by using the LMI toolbox in MATLAB, the solvability of derived conditions depends on not only the size of the delay, but also the probability of the delay-taking values in some intervals. Finally, a numerical example is given to illustrate the feasibility and effectiveness of the proposed method.

PACS numbers: 05.45.Xt, 89.75.−k, 02.50.−r, 02.30.Ks

(Some figures in this article are in colour only in the electronic version)

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1. Introduction

Complex dynamical networks are becoming increasingly important in contemporary society both in science and technology [1–5]. A complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. Examples of complex networks include the Internet, which is a network of routers or domains; the World Wide Web (www), a network of web site; the brain, a network of neurons; food webs; telephone cell graphs and electricity distribution networks, etc. Many of these networks exhibit complexity in the overall topological properties and dynamical properties of the network nodes and the coupled units. The complex nature of complex networks has results in a series of important research problems. In particular, one of the interesting phenomena in complex dynamical networks is the synchronization of all dynamical nodes in the networks [6–14]. The synchronization phenomena are very common and important in real-world networks, such as synchronization phenomena on the Internet, synchronization transfer of digital or analog signals in communication networks and synchronization related to biological neural networks. Hence, synchronization analysis in complex networks is important both in theory and application.

Time delay is ubiquitous in many physical systems due to the finite switching speed of amplifiers, finite signal propagation time in biological networks, finite chemical reaction times, memory effects and so on [15–17]. Moreover, time delay in the interaction may modify drastically the dynamic behavior of the system, such as stability and ergodicity. Therefore, time delays should be modeled in order to simulate realistic networks, and it can exhibit the reality much better. Therefore, the synchronization problem for complex networks with time delays has gained increasing research attention. It is worth pointing out that, among most existing results, the network synchronization problem has been predominantly studied for deterministic complex networks with or without delays, see [3, 7, 8, 18–23] and the references therein. For example, the global synchronization problem for complex networks without delays has been explored in [3, 18, 19]; the network synchronization problem of complex networks with delays or coupling delays has been studied in [7, 8, 20] and the literature [20, 22] has been concerned with the adaptive synchronization problem of some dynamical networks. It is worth mentioning that the time delays can be generally categorized as discrete ones and distributed ones, and the distributed delays have gained particular attention since networks usually have a spatial nature due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths [23]. Recently, synchronization problems for various networks with discrete and/or distributed time delay have extensively studied [4, 24, 25].

Firstly, in real-time systems, complex networks may be subject to networks mode switching, a network sometimes has finite modes that switch from one to another at different times [26, 27]. In [28], the bufferless packet switching of trees and leveled networks has been illustrated to be achievable with certain network topologies. In [27], a sensor network has been shown to have jumping behavior due to the network's working environment and the mobility of sensor node. In [25], exponential synchronization of complex networks with Markovian jump and mixed delay has been studied. In [29] the asymptotic synchronization analysis problem has been investigated for a class of discrete-time stochastic Markovian complex networks with discrete and distributed time delays. In [30, 31], the exponential stability has been studied for delayed recurrent neural networks with Markovian jumping parameters. Secondly, the signal transmission is usually a noisy process brought on by random fluctuations from probabilistic causes and, therefore, stochastic modeling has been of vital importance in many branches of science such as neurotransmitters and packet dropouts. It is often the case that the dynamical behaviors of complex networks are largely affected by the stochastic disturbances. Subsequently, the synchronization problem for stochastic networks has begun to receive some initial research interest. In [4, 9, 10, 32, 33], the synchronization problems have been intensively investigated for delayed complex networks with various kinds of stochastic disturbances, where the criteria ensuring the synchronization among networks have been derived mainly based on the Lyapunov approach that is capable of coping with the different types of time delays. More recently, by using the Lyapunov functional method and Kronecker product technique, the global exponential synchronization has been established in [34] for arrays of coupled identical delayed neural networks with constant and delayed coupling. Thirdly, as is well known, a wide class of practical systems is influenced by additive nonlinear disturbances that are caused by environmental circumstances, the randomly occurring nonlinearities, which have recently received some interest in the literature. For example, in [35], the filtering and control problems for discrete-time systems with stochastic nonlinearities have been thoroughly investigated. Finally, in many practical systems, such as networked control systems, the probability distribution of time delay in the interval is an important characteristic for the network conditions [36]; the probability of the delay appearing in the lower interval is large and long delay happens with a low probability [37–41]. Therefore, the information of probability distribution of the delay should be employed in the model. To the best of the authors' knowledge, no result has been reported for the synchronization stability analysis of complex dynamical networks with Markovian jumping when both the information of variation range of the time delay and the information of variation probability of the time delay in an interval can be observed, which motivates the present study.

Motivated by the above analysis, the synchronization problem is investigated for a class of stochastic complex networks with Markovian jump and probabilistic interval time-varying delays. There are several delay-dependent sufficient conditions under which the complex network is asymptotically synchronized in the mean square by utilizing a new Lyapunov functional and the stochastic analysis techniques; the criteria obtained in this paper are in the form of LMIs whose solution can be easily calculated using the standard numerical software. The solvability of derived conditions depends on not only the size of the delay, but also the probability of the delay-taking values in some intervals. The main novelty of this paper can be summarized as follows. (1) Markovian jumping parameters and mixed time delays are introduced for complex dynamical networks. A primary difference between our model and [25] is that in our model, all matrices contain Markovian jumping parameters, but in [25], the mixed delays are considered that are dependent on the jumping mode. (2) In our paper, the stochastic coupling term and stochastic disturbance are investigated in order to reflect more realistic dynamical behaviors of the complex networks that are affected by noisy environment, which are not considered by [25]. A simulation example is exploited to demonstrate the advantage and applicability to the proposed result.

Notation. The notation used throughout the paper is fairly standard. R^n denotes the n-dimensional Euclidean space and $R^{n\times m}$ is the set of real $n \times m$ matrices. The notation $X > 0$ (respectively, $X < 0$), for $X \in R^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, negative definite). For a real matrix B and two real symmetric matrices A and C of appropriate dimensions, $\begin{bmatrix} A & * \\ B & C \end{bmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. The superscript 'T' stands for matrix transposition. The Kronecker product of matrices $Q \in R^{m\times n}$ and $R \in R^{p\times q}$ is a matrix in $R^{mp\times nq}$ and denoted as $Q \otimes R$. We let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of continuous functions, from $[-\tau, 0]$ to R^n with the norm $|\varphi| = \sup_{-\tau\leqslant\theta\leqslant 0}\|\varphi(\theta)\|$. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be complete probability space. Denote by $L^p_{\mathcal{F}_0}C([-\tau, 0]; R^n)$ the family of all $\mathcal{F}_0$-measurable $C([-\tau, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leqslant \theta \leqslant 0\}$ such that $\sup_{-\tau\leqslant\theta\leqslant 0}\|\xi(\theta)\|^p < \infty$, where $E\{\cdot\}$ stands for the mathematical expectation. In this paper, if not explicitly stated, matrices are assumed to have compatible dimensions.

2. Complex dynamical network model and preliminaries

Let $r(t)(t \geqslant 0)$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ taking values in a finite state space $\mathcal{S} = \{1, 2, \ldots, m\}$ with a generator $\Gamma = (\gamma_{ij})_{m\times m}$ given by

$$P\{r(t+\Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & (i \neq j) \\ 1 + \gamma_{ij}\Delta + o(\Delta) & (i = j) \end{cases} \tag{1}$$

where $\Delta > 0$ and $\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta} = 0$. $\gamma_{ij} \geqslant 0$ is the transition rate from i to j; if $i \neq j$, then $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$.

In this paper, we consider the following stochastic delayed complex dynamical networks by N identical nodes with Markovian jumping parameters and mixed time delays:

$$\begin{aligned} \mathrm{d}x_k(t) = {} & \left[A(r(t))f(x_k(t)) + B(r(t))g(x_k(t-\tau(t))) + C(r(t))\int_{t-\tau}^{t} h(x_k(s))\,\mathrm{d}s\right]\mathrm{d}t \\ & + \sum_{l=1}^{N} G^{(1)}_{kl,r(t)}\Gamma_{1,r(t)}x_l(t)(\mathrm{d}t + \mathrm{d}w_1(t)) + \sum_{l=1}^{N} G^{(2)}_{kl,r(t)}\Gamma_{2,r(t)}x_l(t-\tau(t))(\mathrm{d}t + \mathrm{d}w_2(t)) \\ & + \sigma_k(t, x_k(t), x_k(t-\tau(t)))\,\mathrm{d}w_3(t) \quad (k = 1, 2, \ldots, N) \end{aligned} \tag{2}$$

where $x_k(t) = (x_{k1}(t), x_{k2}(t), \ldots, x_{kn}(t))^T \in R^n$ is the state vector of the kth node. $\{r(t), t > 0\}$ is the continuous-time Markov process which describers the evolution of the mode at time t. $f(\cdot), g(\cdot), h(\cdot) \in R^n$ are sector-bounded continuous nonlinear vector functions. $\Gamma_{1,r(t)}$ and $\Gamma_{2,r(t)} \in R^{n\times n}$ represent the inner coupling between the subsystems at t and $t-\tau(t)$ in mode $r(t)$, respectively. $G^{(1)}_{r(t)} = \big(G^{(1)}_{kl,r(t)}\big)_{N\times N}$ and $G^{(2)}_{r(t)} = \big(G^{(2)}_{kl,r(t)}\big)_{N\times N}$ are the outer-coupling matrices of the networks representing the coupling strength and the topological structure of the complex networks, in which $G^{(q)}_{kl,r(t)}$ is defined as follows: if there exists a connection between the kth node and the lth node $(k \neq l)$, then $G^{(q)}_{kl,r(t)} = G^{(q)}_{kl,r(t)} > 0$, otherwise $G^{(q)}_{kl,r(t)} = G^{(q)}_{kl,r(t)} = 0$ $(k \neq l)$, and the diagonal elements of matrix $G^{(q)}_{r(t)}$ are defined by

$$G^{(q)}_{kk,r(t)} = -\sum_{l=1,k\neq l}^{N} G^{(q)}_{kl,r(t)} \qquad (q = 1, 2; k = 1, 2, \ldots, N). \tag{3}$$

$\sigma_k(\cdot,\cdot,\cdot) : R \times R^n \times R^n \to R^n$ is the noise intensity function vector. The scalar function $\tau(t)$ denotes the time-varying discrete time delay, whereas the scalar $\tau > 0$ describes the distributed time delay. $w_i(t)$ $(i = 1, 2, 3)$ are scalar Brownian motions defined on $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying

$$E\{w_i(t)\} = 0 \qquad E\{(w_i(t))^2\} = t \tag{4}$$

where $w_1(t)$ and $w_2(t)$ represent the coupling strength disturbances while $w_3(t)$ is the system noise. Here $w_i(t)$ $(i = 1, 2, 3)$ is assumed to be mutually independent.

Assumption 1. *There exist constants τ_1 and τ_2, where $0 \leqslant \tau_1 \leqslant \tau_2$, such that either $\tau(t) \in [0, \tau_1]$ or $\tau(t) \in (\tau_1, \tau_2]$. Furthermore, the probability distribution of $\tau(t)$ taking values in $[0, \tau_1]$ and $(\tau_1, \tau_2]$ is known a priori.*

Define the following two sets and functions:

$$\Omega_1 = \{t : \tau(t) \in [0, \tau_1]\} \qquad \Omega_2 = \{t : \tau(t) \in (\tau_1, \tau_2]\} \tag{5}$$

$$\tau_1(t) = \begin{cases} \tau(t) & (t \in \Omega_1) \\ \tau_1 & (t \in \Omega_2) \end{cases} \qquad \tau_2(t) = \begin{cases} \tau(t) & (t \in \Omega_2) \\ \tau_2 & (t \in \Omega_1). \end{cases} \tag{6}$$

From the definitions of Ω_1 and Ω_2, it can be seen that $t \in \Omega_1$ means that the event $\tau(t) \in [0, \tau_1]$ occurs and $t \in \Omega_2$ means that the event $\tau(t) \in (\tau_1, \tau_2]$ occurs. Therefore, a stochastic variable $\beta(t)$ can be defined as

$$\beta(t) = \begin{cases} 1 & (t \in \Omega_1) \\ 0 & (t \in \Omega_2). \end{cases} \tag{7}$$

Assumption 2. *$\beta(t)$ is a Bernoulli distributed sequence with*

$$P\{\beta(t) = 1\} = \beta_0, \qquad P\{\beta(t) = 0\} = 1 - \beta_0. \tag{8}$$

Remark 1. From assumption 1, it can be shown that $E\{\beta(t)\} = \beta_0$ and $E\{(\beta(t) - \beta_0)^2\} = \beta_0(1 - \beta_0)$. Since $P\{\tau(t) \in [0, \tau_1]\} = P\{\beta(t) = 1\}$ and $P\{\tau(t) \in (\tau_1, \tau_2]\} = P\{\beta(t) = 0\}$, β_0 and $1 - \beta_0$ denote the probabilities of $\tau(t)$ taking values in $[0, \tau_1]$ and $(\tau_1, \tau_2]$ respectively.

Remark 2. The introduction of $\beta(t)$ is motivated by [42–45], where the Bernoulli distributed sequence $\beta(t)$ is used to model the randomly varying delay and packet dropout. Different from [42–45], $\beta(t)$ is used in this paper to describe the probability distribution of the time-varying delay taking values in an interval.

By using the new functions $\tau_1(t)$, $\tau_2(t)$ and $\beta(t)$, the system (1) can be rewritten as

$$\begin{aligned} \mathrm{d}x_k(t) = {}& \bigg[A(r(t))f(x_k(t)) + \beta(t)B(r(t))g(x_k(t - \tau_1(t))) \\ & + (1 - \beta(t))B(r(t))g(x_k(t - \tau_2(t))) + C(r(t))\int_{t-\tau}^{t} h(x_k(s))\,\mathrm{d}s\bigg]\mathrm{d}t \\ & + \sum_{l=1}^{N} G^{(1)}_{kl,r(t)}\Gamma_{1,r(t)}x_l(t)(\mathrm{d}t + \mathrm{d}w_1(t)) \\ & + \beta(t)\sum_{l=1}^{N} G^{(2)}_{kl,r(t)}\Gamma_{2,r(t)}x_l(t - \tau_1(t))(\mathrm{d}t + \mathrm{d}w_2(t)) \\ & + (1 - \beta(t))\sum_{l=1}^{N} G^{(2)}_{kl,r(t)}\Gamma_{2,r(t)}x_l(t - \tau_2(t))(\mathrm{d}t + \mathrm{d}w_2(t)) \\ & + [\beta(t)\sigma_k(t, x_k(t), x_k(t - \tau_1(t))) + (1 - \beta(t))\sigma_k(t, x_k(t), x_k(t - \tau_2(t)))]\,\mathrm{d}w_3(t). \end{aligned} \tag{9}$$

Note that the set $\mathcal{S}$ consists of different operation modes of systems (9) for each possible values of $r(t) = i$, $i \in \mathcal{S}$. For the sake of simplicity, we denote the matrix associated with the ith mode by $\Gamma_i = \Gamma(r(t) = i)$; the matrix Γ could be $A, B, C, G^{(1)}, G^{(2)}, \Gamma_1, \Gamma_2$. Therefore, the system (9) could be further rewritten as

$$\begin{aligned} \mathrm{d}x_k(t) = {}& \bigg[A_i f(x_k(t)) + \beta(t)B_i g(x_k(t - \tau_1(t))) + (1 - \beta(t))B_i g(x_k(t - \tau_2(t))) \\ & + C_i\int_{t-\tau}^{t} h(x_k(s))\,\mathrm{d}s\bigg]\mathrm{d}t + \sum_{l=1}^{N} G^{(1)}_{kli}\Gamma_{1i}x_l(t)(\mathrm{d}t + \mathrm{d}w_1(t)) \end{aligned}$$

$$+\beta(t)\sum_{l=1}^{N}G_{kli}^{(2)}\Gamma_{2i}x_l(t-\tau_1(t))(\mathrm{d}t+\mathrm{d}w_2(t))$$
$$+(1-\beta(t))\sum_{l=1}^{N}G_{kli}^{(2)}\Gamma_{2i}x_l(t-\tau_2(t))(\mathrm{d}t+\mathrm{d}w_2(t))$$
$$+[\beta(t)\sigma_k(t,x_k(t),x_k(t-\tau_1(t)))+(1-\beta(t))\sigma_k(t,x_k(t),x_k(t-\tau_2(t)))]\,\mathrm{d}w_3(t). \quad (10)$$

Assumption 3 [35]**.** *For $\forall u, v \in R^n$, the nonlinear functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ satisfy the following sector-bounded condition:*

$$[f(u)-f(v)-F_1(u-v)]^T[f(u)-f(v)-F_2(u-v)] \leqslant 0 \quad (11)$$
$$[g(u)-g(v)-X_1(u-v)]^T[g(u)-g(v)-X_2(u-v)] \leqslant 0 \quad (12)$$
$$[h(u)-h(v)-H_1(u-v)]^T[h(u)-h(v)-H_2(u-v)] \leqslant 0 \quad (13)$$

where F_1, F_2, X_1, X_2, H_1 and H_2 are real constant matrices with $F_2-F_1 \geqslant 0$, $X_2-X_1 \geqslant 0$ and $H_2-H_1 \geqslant 0$.

Remark 3. The nonlinear functions $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ satisfying assumption 3 are said to belong to the sectors $[F_1, F_2]$, $[X_1, X_2]$ and $[H_1, H_2]$ respectively. Note that the sector-bounded nonlinearity of stochastic systems has been studied in [9, 35]. It should be pointed out that such a nonlinear condition is more general than the usual Lipschitz conditions that have been widely used in [4]. By adopting such a presentation, it would be possible to reduce the conservation of the main results caused by quantifying the nonlinear functions via a LMI technique.

Assumption 4. *The noise intensity function vector $\sigma_k : R \times R^n \times R^n \to R^n$ satisfies the Lipschitz condition; there exist some constant matrices Π_1, Π_2, Π_3 and Π_4 of appropriate dimensions such that the following inequalities hold:*

$$\begin{aligned}&[\sigma_k(t,x_k(t),x_k(t-\tau_1(t)))-\sigma_l(t,x_l(t),x_l(t-\tau_1(t))]^T\\ &\qquad\times[\sigma_k(t,x_k(t),x_k(t-\tau_1(t)))-\sigma_l(t,x_l(t),x_l(t-\tau_1(t)))]\\ &\quad\leqslant\|\Pi_1(x_k(t)-x_l(t)\|^2+\|\Pi_2(x_k(t-\tau_1(t))-x_l(t-\tau_1(t))\|^2\end{aligned} \quad (14)$$

$$\begin{aligned}&[\sigma_k(t,x_k(t),x_k(t-\tau_2(t)))-\sigma_l(t,x_l(t),x_l(t-\tau_2(t))]^T\\ &\qquad\times[\sigma_k(t,x_k(t),x_k(t-\tau_2(t)))-\sigma_l(t,x_l(t),x_l(t-\tau_2(t))]\\ &\quad\leqslant\|\Pi_3(x_k(t)-x_l(t)\|^2+\|\Pi_4(x_k(t-\tau_2(t))-x_l(t-\tau_2(t)))\|^2.\end{aligned} \quad (15)$$

By utilizing the Kronecker product of the matrices, the network system (9) can be written in a compact form as

$$\begin{aligned}\mathrm{d}x(t)=\Bigg[&(I_N\otimes A_i)F(x(t))+\beta(t)(I_N\otimes B_i)G(x(t-\tau_1(t)))\\ &+(1-\beta(t))(I_N\otimes B_i)G(x(t-\tau_2(t)))+(I_N\otimes C_i)\int_{t-\tau}^{t}H(x(s))\,\mathrm{d}s\Bigg]\mathrm{d}t\\ &+\big(G_i^{(1)}\otimes\Gamma_{1i}\big)x(t)(\mathrm{d}t+\mathrm{d}w_1(t))+\beta(t)\big(G_i^{(2)}\otimes\Gamma_{2i}\big)x(t-\tau_1(t))\\ &\times(\mathrm{d}t+\mathrm{d}w_2(t))+(1-\beta(t))\big(G_i^{(2)}\otimes\Gamma_{2i}\big)x(t-\tau_2(t))(\mathrm{d}t+\mathrm{d}w_2(t))\\ &+[\beta(t)\sigma^{(1)}(t)+(1-\beta(t))\sigma^{(2)}(t)]\,\mathrm{d}w_3(t)\end{aligned} \quad (16)$$

where

$$x(t) = \left(x_1^T(t), x_2^T(t), \ldots, x_N^T(t)\right)^T$$
$$F(x(t)) = (f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t)))^T$$
$$G(x(t)) = (g^T(x_1(t)), g^T(x_2(t)), \ldots, g^T(x_N(t)))^T$$
$$\int_{t-\tau}^{t} H(x(s))\,\mathrm{d}s = \left(\int_{t-\tau}^{t} h^T(x_1(s))\,\mathrm{d}s, \int_{t-\tau}^{t} h^T(x_2(s))\,\mathrm{d}s, \ldots, \int_{t-\tau}^{t} h^T(x_N(s))\,\mathrm{d}s\right)$$
$$\sigma^{(1)}(t) = \left(\sigma_1^T(t, x_1(t), x_1(t-\tau_1(t))), \quad \sigma_2^T(t, x_2(t), x_2(t-\tau_1(t))), \ldots, \sigma_N^T(t, x_N(t), x_N(t-\tau_1(t)))\right)^T$$
$$\sigma^{(2)}(t) = \left(\sigma_1^T(t, x_1(t), x_1(t-\tau_2(t))), \quad \sigma_2^T(t, x_2(t), x_2(t-\tau_2(t))), \ldots, \sigma_N^T(t, x_N(t), x_N(t-\tau_2(t)))\right)^T.$$

The initial conditions associated with system (9) are given by

$$x_k(s) = \varphi_k(s) \in L_{\mathcal{F}_0}((-\infty, 0], R^n) \qquad (k = 1, 2, \ldots, N) \tag{17}$$

where $L_{\mathcal{F}_0}((-\infty, 0], R^n)$ is the family of all $\mathcal{F}_0$-measurable $\mathcal{C}((-\infty, 0], R^n)$-valued random variables which satisfy $\sup_{-\infty \leqslant s \leqslant 0} E\{\|\varphi_i(s)\|^2\} < \infty$.

Before starting the main results, some definitions and lemmas are introduced here.

Definition 1. *The set $\mathcal{S} = \{x = (x_1(s), x_2(s), \ldots, x_N(s)) : x_k(s) = x_l(s), 1 \leqslant k \leqslant N\}$, is called the synchronization manifold of networks (9):*

Definition 2. *The synchronization manifold $\mathcal{S}$ is said to be globally asymptotically stable in the mean square, if for $\forall \varphi_k(\cdot), \varphi_l(\cdot) \in L_{\mathcal{F}_0}((-\infty, 0], R^n)$, the following holds:*

$$\lim_{t\to\infty} E\{\|x_k(t, \varphi_k(s)) - x_l(t, \varphi_l(s))\|^2\} = 0 \qquad (1 \leqslant k < l \leqslant N). \tag{18}$$

Lemma 1 [46]**.** *By the definition of the Kronecker product, the following properties can be proved:*

$$(\alpha A) \otimes B = A \otimes (\alpha B)$$
$$(A + B) \otimes C = A \otimes C + B \otimes C$$
$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$
$$(A \otimes B)^T = A^T \otimes B^T.$$

Lemma 2. *Let $\mathcal{U} = (\alpha_{ij})_{N\times N}$, $P \in R^{n\times n}$, $x = \left(x_1^T, x_2^T, \ldots, x_N^T\right)^T$, where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ and $x = \left(x_1^T, x_2^T, \ldots, x_N^T\right)^T$, where $y_i = (y_{i1}, y_{i2}, \ldots, y_{in})^T \in R^n$; if $\mathcal{U} = \mathcal{U}^T$ and in each row sum of $\mathcal{U}$ is zero, then*

$$x^T(\mathcal{U} \otimes P)y = -\sum_{1\leqslant i<j\leqslant N} \alpha_{ij}(x_i - x_j)^T P(y_i - y_j). \tag{19}$$

Lemma 3. *Suppose $\tau_1 \leqslant \tau(t) \leqslant \tau_2$, Q_i $(i = 1, 2, 3)$ are some constant matrices with appropriate dimensions; then*

$$Q_1 + (\tau_2 - \tau(t))Q_2 + (\tau(t) - \tau_1)Q_3 < 0$$

if and only if the following inequalities hold:

$$Q_1 + (\tau_2 - \tau_1)Q_2 < 0$$
$$Q_1 + (\tau_2 - \tau_1)Q_3 < 0.$$

Lemma 4. *Q_{1i}, Q_{2i} $(i = 1, 2)$ and Q are constant matrices of appropriate dimensions, $\tau_i(t)$ $(i = 1, 2)$ satisfies $0 \leqslant \tau_1(t) \leqslant \tau_1 \leqslant \tau_2(t) \leqslant \tau_2$; then*

$$[\tau_1(t)Q_{11} + (\tau_1 - \tau_1(t))Q_{21}] + [(\tau_2(t) - \tau_1)Q_{12} + (\tau_2 - \tau_2(t))Q_{22}] + Q < 0$$

if and only if

$$\tau_1 Q_{11} + (\tau_2 - \tau_1)Q_{12} + Q < 0$$
$$\tau_1 Q_{11} + (\tau_2 - \tau_1)Q_{22} + Q < 0$$
$$\tau_1 Q_{21} + (\tau_2 - \tau_1)Q_{12} + Q < 0$$
$$\tau_1 Q_{21} + (\tau_2 - \tau_1)Q_{22} + Q < 0.$$

Lemma 5. *Let $f(\cdot)$ be a nonnegative function defined on $[0, +\infty)$; if $f(\cdot)$ is Lebesgue integrable and is uniformly continuous on $[0, +\infty)$, then $\lim_{t\to+\infty} f(t) = 0$.*

3. Main results and proofs

In this section, we present the synchronization criteria for the delayed complex networks with stochastic disturbances.

We rewrite (16) as

$$\begin{aligned} \mathrm{d}x(t) = \Big\{&(I_N \otimes A_i)F(x(t)) + \beta_0(I_N \otimes B_i)G(x(t - \tau_1(t))) \\ &+ (1 - \beta_0)(I_N \otimes B_i)G(x(t - \tau_2(t))) + (\beta(t) - \beta_0)[(I_N \otimes B_i)G(x(t - \tau_1(t))) \\ &- (I_N \otimes B_i)G(x(t - \tau_2(t)))] + (I_N \otimes C_i)\int_{t-\tau}^{t} H(x(s))\,\mathrm{d}s + \big(G_i^{(1)} \otimes \Gamma_{1i}\big)x(t) \\ &+ \beta_0\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_1(t)) + (1 - \beta_0)\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_2(t)) \\ &+ (\beta(t) - \beta_0)\big[\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_1(t)) - \big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_2(t))\big]\Big\}\,\mathrm{d}t \\ &+ \big(G_i^{(1)} \otimes \Gamma_{1i}\big)x(t)\,\mathrm{d}w_1(t) + \beta(t)\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_1(t))\,\mathrm{d}w_2(t) \\ &+ (1 - \beta(t))\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_2(t))\,\mathrm{d}w_2(t) \\ &+ [\beta(t)\sigma^{(1)}(t) + (1 - \beta(t))\sigma^{(2)}(t)]\,\mathrm{d}w_3(t). \end{aligned} \tag{20}$$

Define

$$y(t) = \mathcal{A}\xi(t) \tag{21}$$

where

$$\begin{aligned} \mathcal{A} = \big[&G_i^{(1)} \otimes \Gamma_{1i} \quad \beta_0\big(G_i^{(2)} \otimes \Gamma_{2i}\big) \quad 0 \quad (1 - \beta_0)\big(G_i^{(2)} \otimes \Gamma_{2i}\big) \quad 0 \quad I_N \otimes A_i \quad \beta_0(I_N \otimes B_i) \\ &(1 - \beta_0)(I_N \otimes B_i) \quad 0 \quad I_N \otimes C_i \quad 0\big] \end{aligned}$$

$$\begin{aligned} \xi^T(t) = \big[&x^T(t) \quad x^T(t - \tau_1(t)) \quad x^T(t - \tau_1) \quad x^T(t - \tau_2(t)) \quad x^T(t - \tau_2) \quad F^T(x(t)) \\ &G^T(x(t - \tau_1(t))) \quad G^T(x(t - \tau_2(t))) \quad H^T(x(t)) \quad \textstyle\int_{t-\tau}^{t} H^T(x(s))\,\mathrm{d}s \quad y^T(t)\big]; \end{aligned}$$

then system (20) can be expressed as

$$\mathrm{d}x(t) = (y(t) + (\beta(t) - \beta_0)\mathcal{B}\xi(t))\,\mathrm{d}t + \bar{\sigma}(t)\,\mathrm{d}w(t) \tag{22}$$

where

$$\mathcal{B} = \big[0 \quad \big(G_i^{(2)} \otimes \Gamma_{2i}\big) \quad 0 \quad -\big(G_i^{(2)} \otimes \Gamma_{2i}\big) \quad 0 \quad 0 \quad (I_N \otimes B_i) \quad -(I_N \otimes B_i) \quad 0 \quad 0 \quad 0\big]$$

$$\begin{aligned} \bar{\sigma}(t) = \big[&\big(G_i^{(1)} \otimes \Gamma_{1i}\big)x(t) \quad \beta(t)\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_1(t)) + (1 - \beta(t))\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t - \tau_2(t)) \\ &\beta(t)\sigma^{(1)}(t) + (1 - \beta(t))\sigma^{(2)}(t)\big] \end{aligned}$$

$$\mathrm{d}w^T(t) = \big[\mathrm{d}w_1^T(t) \quad \mathrm{d}w_2^T(t) \quad \mathrm{d}w_3^T(t)\big].$$

Theorem 1. *For given scalars $\tau_2 > \tau_1 > 0, \tau > 0$ and $\beta_0 > 0$, the stochastic delayed networks (9) with mixed time delays and Markov switching are asymptotically synchronized in the mean square, if there exist appropriate dimensional matrices $P_i > 0\,(i = 1, 2, \ldots, m), Q_i > 0, Z_i > 0\,(i = 1, 2), R_j > 0\,(j = 1, 2, 3)$ and $Y_i, M_i, T_i, W_i\ (i = 1, 2)$ and $S_j\ (j = 1, 2, 3, 4)$ and positive scalars $\lambda_1 > 0, \lambda_2 > 0, \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$ such that the following linear matrix inequalities hold for all $1 \leqslant k < l \leqslant N$:*

$$\begin{bmatrix} \Sigma_{11i} & * \\ \Sigma_{21}^l & \Sigma_{22} \end{bmatrix} < 0 \tag{23}$$

$$P_i - \lambda_i I < 0 \qquad (i = 1, 2, \ldots, m; l = 1, 2, 3, 4) \tag{24}$$

where

$$\Sigma_{11i} = \begin{bmatrix}
\Theta_{11i} & * & * & * & * & * & * & * & * & * & * \\
\Theta_{21i} & \Theta_{22i} & * & * & * & * & * & * & * & * & * \\
0 & \Theta_{32i} & \Theta_{33i} & * & * & * & * & * & * & * & * \\
\Theta_{41i} & \Theta_{42i} & \Theta_{43i} & \Theta_{44i} & * & * & * & * & * & * & * \\
0 & 0 & 0 & \Theta_{54i} & \Theta_{55i} & * & * & * & * & * & * \\
\Theta_{61i} & A_i^T S_2^T & 0 & A_i^T S_3^T & 0 & -2\alpha_1 I & * & * & * & * & * \\
\Theta_{71i} & \Theta_{72i} & 0 & \Theta_{74i} & 0 & 0 & \Theta_{77i} & * & * & * & * \\
\Theta_{81i} & \Theta_{82i} & 0 & \Theta_{84i} & 0 & 0 & \Theta_{87i} & \Theta_{88i} & * & * & * \\
\Theta_{91i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Theta_{99i} & * & * \\
C_i^T S_1^T & C_i^T S_2^T & 0 & C_i^T S_3^T & 0 & 0 & 0 & 0 & 0 & -R_3 & * \\
\Theta_{11,1i} & \Theta_{11,2i} & 0 & \Theta_{11,4i} & 0 & S_4 A_i & \beta_0 S_4 B_i & \Theta_{11,8i} & 0 & S_4 C_i & \Theta_{11,11i}
\end{bmatrix}$$

$$\Sigma_{21}^1 = \begin{bmatrix} \tau_1 M^T \\ \tau_1 M^T \\ (\tau_2 - \tau_1) W^T \\ (\tau_2 - \tau_1) W^T \end{bmatrix} \quad \Sigma_{21}^2 = \begin{bmatrix} \tau_1 M^T \\ \tau_1 M^T \\ (\tau_2 - \tau_1) T^T \\ (\tau_2 - \tau_1) T^T \end{bmatrix} \quad \Sigma_{21}^3 = \begin{bmatrix} \tau_1 Y^T \\ \tau_1 Y^T \\ (\tau_2 - \tau_1) W^T \\ (\tau_2 - \tau_1) W^T \end{bmatrix} \quad \Sigma_{21}^4 = \begin{bmatrix} \tau_1 Y^T \\ \tau_1 Y^T \\ (\tau_2 - \tau_1) T^T \\ (\tau_2 - \tau_1) T^T \end{bmatrix}$$

$$\Sigma_{22} = \operatorname{diag}\{-\tau_1 R_1 - \tau_1 Z_1 - (\tau_2 - \tau_1) R_2 - (\tau_2 - \tau_1) Z_2\}$$

$$\begin{aligned}
\Theta_{11i} &= \sum_{j=1}^{m} \gamma_{ij} P_j - N G_{kli}^{(1,1)} \Gamma_{1i}^T P_i \Gamma_{1i} + 2\beta_0 \lambda_i \Pi_1^T \Pi_1 + 2(1 - \beta_0) \lambda_i \Pi_3^T \Pi_3 + Q_1 + Q_2 \\
&\quad + Y_1 + Y_1^T - N G_{kli}^{(1)} S_1 \Gamma_{1i} - N G_{kli}^{(1)} \Gamma_{1i}^T S_1^T - \alpha_1 \big(F_1^T F_2 + F_2^T F_1\big) \\
&\quad - 2\alpha_2 \big(X_1^T X_2 + X_2^T X_1\big) - \alpha_3 \big(H_1^T H_2 + H_2^T H_1\big) \\
\Theta_{21i} &= Y_2 - Y_1^T - \beta_0 N G_{kli}^{(2)} \Gamma_{2i}^T S_1^T - N G_{kli}^{(1)} S_2 \Gamma_{1i} \\
\Theta_{22i} &= -2N\beta_0 G_{kli}^{(2,2)} \Gamma_{2i}^T P_i \Gamma_{2i} + 2\beta_0 \lambda_i \Pi_2^T \Pi_2 - \beta_0 (1 - \beta_0) N \tau_1 G_{kli}^{(2,2)} \Gamma_{2i}^T Z_1 \Gamma_{2i} \\
&\quad - \beta_0 (1 - \beta_0) N (\tau_2 - \tau_1) G_{kli}^{(2,2)} \Gamma_{2i}^T Z_2 \Gamma_{2i} - Y_2 - Y_2^T + M_1 + M_1^T \\
&\quad - \beta_0 N G_{kli}^{(2)} S_2 \Gamma_{2i} - \beta_0 N G_{kli}^{(2)} \Gamma_{2i}^T S_2^T \\
\Theta_{32i} &= M_2 - M_1^T \\
\Theta_{33i} &= -Q_1 - M_2 - M_2^T + T_1 + T_1^T \\
\Theta_{41i} &= -(1 - \beta_0) N G_{kli}^{(2)} \Gamma_{2i}^T S_1^T - N G_{kli}^{(1)} S_3 \Gamma_{1i} \\
\Theta_{42i} &= \beta_0 (1 - \beta_0) N \tau_1 G_{kli}^{(2,2)} \Gamma_{2i}^T Z_1 \Gamma_{2i} + \beta_0 (1 - \beta_0) N (\tau_2 - \tau_1) G_{kli}^{(2,2)} \Gamma_{2i}^T Z_2 \Gamma_{2i} \\
&\quad - (1 - \beta_0) N G_{kli}^{(2)} \Gamma_{2i}^T S_2^T - \beta_0 N G_{kli}^{(2)} S_3 \Gamma_{2i}
\end{aligned}$$

$$
\begin{aligned}
\Theta_{43i} &= T_2 - T_1^T \\
\Theta_{44i} &= -2N(1-\beta_0)G_{kli}^{(2,2)}\Gamma_{2i}^T P_i \Gamma_{2i} + 2(1-\beta_0)\lambda_i \Pi_4^T \Pi_4 - \beta_0(1-\beta_0)N\tau_1 G_{kli}^{(2,2)}\Gamma_{2i}^T Z_1 \Gamma_{2i} \\
&\quad - \beta_0(1-\beta_0)N(\tau_2-\tau_1)G_{kli}^{(2,2)}\Gamma_{2i}^T Z_2\Gamma_{2i} - T_2 - T_2^T + W_1 + W_1^T \\
&\quad - (1-\beta_0)NG_{kli}^{(2)}S_3\Gamma_{2i} - (1-\beta_0)NG_{kli}^{(2)}\Gamma_{2i}^T S_3^T \\
\Theta_{54i} &= W_2 - W_1^T \\
\Theta_{55i} &= -Q_2 - W_2 - W_2^T \\
\Theta_{61i} &= A_i^T S_1^T + \alpha_1(F_1+F_2) \\
\Theta_{71i} &= \beta_0 B_i^T S_1^T + \alpha_2(X_1+X_2) \\
\Theta_{72i} &= -\beta_0(1-\beta_0)N\tau_1 G_{kli}^{(2,2)}\Gamma_{2i}^T Z_1\Gamma_{2i} - \beta_0(1-\beta_0)N(\tau_2-\tau_1)G_{kli}^{(2,2)}\Gamma_{2i}^T Z_2\Gamma_{2i} + \beta_0 B_i^T S_2^T \\
\Theta_{74i} &= \beta_0(1-\beta_0)N\tau_1 G_{kli}^{(2,2)}\Gamma_{2i}^T Z_1\Gamma_{2i} + \beta_0(1-\beta_0)N(\tau_2-\tau_1)G_{kli}^{(2,2)}\Gamma_{2i}^T Z_2\Gamma_{2i} + \beta_0 B_i^T S_3^T \\
\Theta_{77i} &= \beta_0(1-\beta_0)\tau_1 B_i^T Z_1 B_i + \beta_0(1-\beta_0)(\tau_2-\tau_1)B_i^T Z_2 B_i - 2\alpha_2 I \\
\Theta_{81i} &= (1-\beta_0)B_i^T S_1^T + \alpha_2(X_1+X_2) \\
\Theta_{82i} &= \beta_0(1-\beta_0)N\tau_1 B_i^T Z_1\Gamma_{2i} + \beta_0(1-\beta_0)N(\tau_2-\tau_1)B_i^T Z_2\Gamma_{2i} + (1-\beta_0)B_i^T S_2^T \\
\Theta_{84i} &= -\beta_0(1-\beta_0)N\tau_1 B_i^T Z_1\Gamma_{2i} - \beta_0(1-\beta_0)N(\tau_2-\tau_1)B_i^T Z_2\Gamma_{2i} + (1-\beta_0)B_i^T S_3^T \\
\Theta_{87i} &= -\beta_0(1-\beta_0)\tau_1 B_i^T Z_1 B_i - \beta_0(1-\beta_0)(\tau_2-\tau_1)B_i^T Z_2 B_i \\
\Theta_{88i} &= \beta_0(1-\beta_0)\tau_1 B_i^T Z_1 B_i + \beta_0(1-\beta_0)(\tau_2-\tau_1)B_i^T Z_2 B_i - 2\alpha_2 I \\
\Theta_{91i} &= \alpha_3(H_1+H_2) \\
\Theta_{99i} &= \tau^2 R_3 - 2\alpha_3 I \\
\Theta_{11,1i} &= P_i - NG_{kli}^{(1)}S_4\Gamma_{1i} - S_1^T \\
\Theta_{11,2i} &= -\beta_0 NG_{kli}^{(2)}S_4\Gamma_{2i} - S_2^T \\
\Theta_{11,4i} &= -(1-\beta_0)NG_{kli}^{(2)}S_4\Gamma_{2i} - S_3^T \\
\Theta_{11,8i} &= (1-\beta_0)S_4 B_i \\
\Theta_{11,11i} &= \tau_1 R_1 + (\tau_2-\tau_1)R_2 - S_4 - S_4^T \\
Y^T &= \begin{bmatrix} Y_1^T & Y_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
M^T &= \begin{bmatrix} 0 & M_1^T & M_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
T^T &= \begin{bmatrix} 0 & 0 & T_1^T & T_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
W^T &= \begin{bmatrix} 0 & 0 & 0 & W_1^T & W_2^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
S^T &= \begin{bmatrix} S_1^T & S_2^T & 0 & S_3^T & 0 & 0 & 0 & 0 & 0 & 0 & S_4^T \end{bmatrix}.
\end{aligned}
$$

Proof. For presentation convenience, let $x_{kl}(t) = x_k(t) - x_l(t)$, $y_{kl}(t) = y_k(t) - y_l(t)$, $f_{kl}(t) = f(x_k(t)) - f(x_l(t))$, $g_{kl}(t) = g(x_k(t)) - g(x_l(t))$, $h_{kl}(t) = h(x_k(t)) - h(x_l(t))$, $\int_{t-\tau}^{t} h_{kl}(s)\,\mathrm{d}s = \int_{t-\tau}^{t}[h(x_k(s)) - h(x_l(s))]\,\mathrm{d}s$, $(\mathcal{B}\xi)_{kl}(t) = (\mathcal{B}\xi)_k(t) - (\mathcal{B}\xi)_l(t)$, $\xi_{kl}(t) = \xi_k(t) - \xi_l(t)$. Denote by $C^{2,1}(R_+ \times R \times \mathcal{S};\, R^n)$ the family of nonnegative functions $V(t, x(t), i)$ on $R_+ \times R \times \mathcal{S}$ which are once differentiable with respect to the first variable t and twice differentiable with respect to the second variable $x(t)$. Construct a Lyapunov–Krasovskii functional candidate as

$$V(t, x(t), i) = V_1(t, x(t), i) + V_2(t, x(t), i) + V_3(t, x(t), i) + V_4(t, x(t), i) \qquad (25)$$

where

$$V_1(t, x(t), i) = x^T(t)(\mathcal{U} \otimes P_i)x(t)$$

$$V_2(t, x(t), i) = \int_{t-\tau_1}^{t} x^T(s)(\mathcal{U} \otimes Q_1)x(s)\,\mathrm{d}s + \int_{t-\tau_2}^{t} x^T(s)(\mathcal{U} \otimes Q_2)x(s)\,\mathrm{d}s$$

$$V_3(t, x(t), i) = \int_{t-\tau_1}^{t}\int_{s}^{t} y^T(v)(\mathcal{U} \otimes R_1)y(v)\,\mathrm{d}v\,\mathrm{d}s + \int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t} y^T(v)(\mathcal{U} \otimes R_2)y(v)\,\mathrm{d}v\,\mathrm{d}s$$
$$+ \tau\int_{t-\tau}^{t}\int_{s}^{t} H^T(x(v))(\mathcal{U} \otimes R_3)H(x(v))\,\mathrm{d}v\,\mathrm{d}s$$

$$V_4(t, x(t), i) = \beta_0(1-\beta_0)\int_{t-\tau_1}^{t}\int_{s}^{t} \xi^T(v)\mathcal{B}^T(\mathcal{U} \otimes Z_1)\mathcal{B}\xi(v)\,\mathrm{d}v\,\mathrm{d}s$$
$$+ \beta_0(1-\beta_0)\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t} \xi^T(v)\mathcal{B}^T(\mathcal{U} \otimes Z_2)\mathcal{B}\xi(v)\,\mathrm{d}v\,\mathrm{d}s$$

where P_i $(i = 1, 2, \ldots, m)$, $Q_1, Q_2, R_1, R_2, R_3, Z_l, Z_2 > 0$ and

$$\mathcal{U} = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix}_{N\times N}.$$

The infinitesimal operator $\mathcal{L}$ of $V(t, x(t), i)$ is defined as follows [47]:

$$\mathcal{L}V(t, x(t), i) = \lim_{\Delta\to 0^+}\frac{1}{\Delta}\sup\{E(V(t+\Delta, x(t+\Delta),$$
$$r(t+\Delta))|t, x(t), r(t) = i) - V(t, x(t), r(t) = i)\}. \qquad (26)$$

From (25) and (26), we can obtain

$$\mathcal{L}V(t, x(t), i) = \mathcal{L}V_1(t, x(t), i) + \mathcal{L}V_2(t, x(t), i) + \mathcal{L}V_3(t, x(t), i) + \mathcal{L}V_4(t, x(t), i). \qquad (27)$$

Calculating the time derivative of V along the trajectories of (22), we have

$$\mathcal{L}V_1(t, x(t), i) = 2x^T(t)(\mathcal{U} \otimes P_i)y(t) + \sum_{j=1}^{m}\gamma_{ij}V_1(t, x(t), j) + E\{\bar{\sigma}^T(t)(\mathcal{U} \otimes P_i)\bar{\sigma}(t)\}$$
$$= 2x^T(t)(\mathcal{U} \otimes P_i)y(t) + \sum_{j=1}^{m}\gamma_{ij}x^T(t)(\mathcal{U} \otimes P_i)x(t)$$
$$+ x^T(t)\big(G_i^{(1)} \otimes \Gamma_{1i}\big)^T(\mathcal{U} \otimes P_i)\big(G_i^{(1)} \otimes \Gamma_{1i}\big)x(t)$$
$$+ E\big\{\big[\beta(t)\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t-\tau_1(t))$$
$$+ (1-\beta(t))\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t-\tau_2(t))\big]^T(\mathcal{U} \otimes P_i)$$
$$\times\big[\beta(t)\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t-\tau_1(t)) + (1-\beta(t))\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t-\tau_2(t))\big]\big\}$$
$$+ E\{[\beta(t)\sigma^{(1)}(t) + (1-\beta(t))\sigma^{(2)}(t)]^T(\mathcal{U} \otimes P_i)$$
$$\times[\beta(t)\sigma^{(1)}(t) + (1-\beta(t))\sigma^{(2)}(t)]\}$$
$$\leqslant 2x^T(t)(\mathcal{U} \otimes P_i)y(t) + \sum_{j=1}^{m}\gamma_{ij}x^T(t)(\mathcal{U} \otimes P_i)x(t)$$
$$+ x^T(t)\big(G_i^{(1)} \otimes \Gamma_{1i}\big)^T(\mathcal{U} \otimes P_i)\big(G_i^{(1)} \otimes \Gamma_{1i}\big)x(t)$$
$$+ 2\beta_0 x\big(t-\tau_1^T(t)\big)\big(G_i^{(2)} \otimes \Gamma_{2i}\big)^T(\mathcal{U} \otimes P_i)\big(G_i^{(2)} \otimes \Gamma_{2i}\big)x(t-\tau_1(t))$$

$$+ 2(1-\beta_0)x^T(t-\tau_2(t))\big(G_i^{(2)}\otimes\Gamma_{2i}\big)^T(\mathcal{U}\otimes P_i)\big(G_i^{(2)}\otimes\Gamma_{2i}\big)x(t-\tau_2(t))$$
$$+ 2\beta_0\sigma^{(1)T}(t)(\mathcal{U}\otimes P_i)\sigma^{(1)}(t) + 2(1-\beta_0)\sigma^{(2)T}(t)(\mathcal{U}\otimes P_i)\sigma^{(2)}(t) \tag{28}$$

$$\mathcal{L}V_2(t,x(t),i) = x^T(t)(\mathcal{U}\otimes Q_1 + \mathcal{U}\otimes Q_2)x(t) - x^T(t-\tau_1)(\mathcal{U}\otimes Q_1)x(t-\tau_1) - x^T(t-\tau_2)(\mathcal{U}\otimes Q_2)x(t-\tau_2) \tag{29}$$

$$\mathcal{L}V_3(t,x(t),i) = y^T(t)[\tau_1(\mathcal{U}\otimes R_1) + (\tau_2-\tau_1)(\mathcal{U}\otimes R_2)]y(t) + \tau^2 H^T(t)(\mathcal{U}\otimes R_3)H(t) - \int_{t-\tau_1}^{t} y^T(s)(\mathcal{U}\otimes R_1)y(s)\,\mathrm{d}s - \int_{t-\tau_2}^{t-\tau_1} y^T(s)(\mathcal{U}\otimes R_2)y(s)\,\mathrm{d}s - \tau\int_{t-\tau}^{t} H^T(s)(\mathcal{U}\otimes R_3)H(s)\,\mathrm{d}s \tag{30}$$

$$\leqslant y^T(t)[\tau_1(\mathcal{U}\otimes R_1) + (\tau_2-\tau_1)(\mathcal{U}\otimes R_2)]y(t) + \tau^2 H^T(t)(\mathcal{U}\otimes R_3)H(t) - \int_{t-\tau_1}^{t} y^T(s)(\mathcal{U}\otimes R_1)y(s)\,\mathrm{d}s - \int_{t-\tau_2}^{t-\tau_1} y^T(s)(\mathcal{U}\otimes R_2)y(s)\,\mathrm{d}s - \left[\int_{t-\tau}^{t} H(s)\,\mathrm{d}s\right]^T(\mathcal{U}\otimes R_3)\left[\int_{t-\tau}^{t} H(s)\,\mathrm{d}s\right] \tag{31}$$

$$\mathcal{L}V_4(t,x(t),i) = \beta_0(1-\beta_0)\xi^T(t)\mathcal{B}^T[\tau_1(\mathcal{U}\otimes Z_1) + (\tau_2-\tau_1)(\mathcal{U}\otimes Z_2]\mathcal{B}\xi(t) - \int_{t-\tau_1}^{t}\xi^T(s)\mathcal{B}^T(\mathcal{U}\otimes Z_1)\mathcal{B}\xi(s)\,\mathrm{d}s - \int_{t-\tau_2}^{t-\tau_1}\xi^T(s)\mathcal{B}^T(\mathcal{U}\otimes Z_2)\mathcal{B}\xi(s)\,\mathrm{d}s. \tag{32}$$

Employing the free matrix method [48–50], we have

$$2\xi^T(t)(\mathcal{U}\otimes Y)\left[x(t) - x(t-\tau_1(t)) - \int_{t-\tau_1(t)}^{t}\mathrm{d}x(s)\right] = 0 \tag{33}$$

$$2\xi^T(t)(\mathcal{U}\otimes M)\left[x(t-\tau_1(t)) - x(t-\tau_1) - \int_{t-\tau_1}^{t-\tau_1(t)}\mathrm{d}x(s)\right] = 0 \tag{34}$$

$$2\xi^T(t)(\mathcal{U}\otimes T)\left[x(t-\tau_1) - x(t-\tau_2(t)) - \int_{t-\tau_2(t)}^{t-\tau_1}\mathrm{d}x(s)\right] = 0 \tag{35}$$

$$2\xi^T(t)(\mathcal{U}\otimes W)\left[x(t-\tau_2(t)) - x(t-\tau_2) - \int_{t-\tau_2}^{t-\tau_2(t)}\mathrm{d}x(s)\right] = 0 \tag{36}$$

$$2\xi^T(t)(\mathcal{U}\otimes S)[\mathcal{A}\xi(t) - y(t)] = 0. \tag{37}$$

It can be shown from (33)–(36), there exist $R_1 > 0$, $R_2 > 0$, $Z_1 > 0$, $Z_2 > 0$, such that

$$-2\xi^T(t)(\mathcal{U}\otimes Y)\int_{t-\tau_1(t)}^{t}\mathrm{d}x(s)$$
$$= -2\xi^T(t)(\mathcal{U}\otimes Y)\int_{t-\tau_1(t)}^{t}[y(s) + (\beta(s)-\beta_0)\mathcal{B}\xi(s)]\,\mathrm{d}s - 2\xi^T(t)(\mathcal{U}\otimes Y)\int_{t-\tau_1(t)}^{t}\bar{\sigma}(s)\,\mathrm{d}w(s)$$
$$\leqslant \tau_1(t)\xi^T(t)(\mathcal{U}\otimes Y)[(\mathcal{U}\otimes R_1)^{-1} + (\mathcal{U}\otimes Z_1)^{-1}](\mathcal{U}\otimes Y)^T\xi(t)$$

$$
\begin{aligned}
&+\int_{t-\tau_1(t)}^{t} y^T(s)(\mathcal{U}\otimes R_1)y(s)\,\mathrm{d}s\\
&+\int_{t-\tau_1(t)}^{t} (\beta(s)-\beta_0)^2(\mathcal{B}\xi(s))^T(\mathcal{U}\otimes Z_1)(\mathcal{B}\xi(s))\,\mathrm{d}s\\
&-2\xi^T(t)(\mathcal{U}\otimes Y)\int_{t-\tau_1(t)}^{t}\bar{\sigma}(s)\,\mathrm{d}w(s).
\end{aligned}
\tag{38}
$$

Similarly, we have

$$
\begin{aligned}
-2\xi^T(t)(\mathcal{U}\otimes M)\int_{t-\tau_1}^{t-\tau_1(t)}\mathrm{d}x(s)
&\leqslant (\tau_1-\tau_1(t))\xi^T(t)(\mathcal{U}\otimes M)[(\mathcal{U}\otimes R_1)^{-1}+(\mathcal{U}\otimes Z_1)^{-1}](\mathcal{U}\otimes M)^T\xi(t)\\
&\quad+\int_{t-\tau_1}^{t-\tau_1(t)} y^T(s)(\mathcal{U}\otimes R_1)y(s)\,\mathrm{d}s\\
&\quad+\int_{t-\tau_1}^{t-\tau_1(t)} (\beta(s)-\beta_0)^2(\mathcal{B}\xi(s))^T(\mathcal{U}\otimes Z_1)(\mathcal{B}\xi(s))\,\mathrm{d}s\\
&\quad-2\xi^T(t)(\mathcal{U}\otimes Y)\int_{t-\tau_1}^{t-\tau_1(t)}\bar{\sigma}(s)\,\mathrm{d}w(s)
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
-2\xi^T(t)(\mathcal{U}\otimes T)\int_{t-\tau_2(t)}^{t-\tau_1}\mathrm{d}x(s)
&\leqslant (\tau_2(t)-\tau_1)\xi^T(t)(\mathcal{U}\otimes T)[(\mathcal{U}\otimes R_2)^{-1}+(\mathcal{U}\otimes Z_2)^{-1}](\mathcal{U}\otimes T)^T\xi(t)\\
&\quad+\int_{t-\tau_2(t)}^{t-\tau_1} y^T(s)(\mathcal{U}\otimes R_2)y(s)\,\mathrm{d}s\\
&\quad+\int_{t-\tau_2(t)}^{t-\tau_1} (\beta(s)-\beta_0)^2(\mathcal{B}\xi(s))^T(\mathcal{U}\otimes Z_2)(\mathcal{B}\xi(s))\,\mathrm{d}s\\
&\quad-2\xi^T(t)(\mathcal{U}\otimes T)\int_{t-\tau_2(t)}^{t-\tau_1}\bar{\sigma}(s)\,\mathrm{d}w(s)
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
-2\xi^T(t)(\mathcal{U}\otimes W)\int_{t-\tau_2}^{t-\tau_2(t)}\mathrm{d}x(s)
&\leqslant (\tau_2-\tau_2(t))\xi^T(t)(\mathcal{U}\otimes W)[(\mathcal{U}\otimes R_2)^{-1}+(\mathcal{U}\otimes Z_2)^{-1}](\mathcal{U}\otimes W)^T\xi(t)\\
&\quad+\int_{t-\tau_2}^{t-\tau_2(t)} y^T(s)(\mathcal{U}\otimes R_2)y(s)\,\mathrm{d}s\\
&\quad+\int_{t-\tau_2}^{t-\tau_2(t)} (\beta(s)-\beta_0)^2(\mathcal{B}\xi(s))^T(\mathcal{U}\otimes Z_2)(\mathcal{B}\xi(s))\,\mathrm{d}s\\
&\quad-2\xi^T(t)(\mathcal{U}\otimes W)\int_{t-\tau_2}^{t-\tau_2(t)}\bar{\sigma}(s)\,\mathrm{d}w(s).
\end{aligned}
\tag{41}
$$

According to assumption 4 and condition (24), we have

$$
\begin{aligned}
&[\sigma_k(t,x_k(t),x_k(t-\tau_1(t)))-\sigma_l(t,x_l(t),x_l(t-\tau_1(t))]^T P_i\\
&\qquad\times[\sigma_k(t,x_k(t),x_k(t-\tau_1(t)))-\sigma_l(t,x_l(t),x_l(t-\tau_1(t))]\\
&\qquad\leqslant\lambda_i\{\|\Pi_1(x_k(t)-x_l(t))\|^2+\|\Pi_2(x_k(t-\tau_1(t))-x_l(t-\tau_1(t)))\|^2\}
\end{aligned}
\tag{42}
$$

$$\begin{aligned}&[\sigma_k(t, x_k(t), x_k(t-\tau_2(t))) - \sigma_l(t, x_l(t), x_l(t-\tau_2(t))]^T P_i\\ &\qquad\times [\sigma_k(t, x_k(t), x_k(t-\tau_2(t))) - \sigma_l(t, x_l(t), x_l(t-\tau_2(t))]\\ &\qquad\leqslant \lambda_i\{\|\Pi_3(x_k(t)-x_l(t))\|^2 + \|\Pi_4(x_k(t-\tau_2(t)) - x_l(t-\tau_2(t)))\|^2\}.\end{aligned} \tag{43}$$

Moreover, from assumption 3, for $\forall\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$, it can be derived that

$$\alpha_1 \begin{bmatrix} x_{kl}(t) \\ f_{kl}(t)\end{bmatrix}^T \begin{bmatrix} F_1^T F_2 + F_2^T F_1 & -F_1^T - F_2^T \\ -F_1 - F_2 & 2I\end{bmatrix}\begin{bmatrix} x_{kl}(t) \\ f_{kl}(t)\end{bmatrix} \leqslant 0 \tag{44}$$

$$\alpha_2 \begin{bmatrix} x_{kl}(t) \\ g_{kl}(t)\end{bmatrix}^T \begin{bmatrix} X_1^T X_2 + X_2^T X_1 & -X_1^T - X_2^T \\ -X_1 - X_2 & 2I\end{bmatrix}\begin{bmatrix} x_{kl}(t) \\ g_{kl}(t)\end{bmatrix} \leqslant 0 \tag{45}$$

$$\alpha_3 \begin{bmatrix} x_{kl}(t) \\ h_{kl}(t)\end{bmatrix}^T \begin{bmatrix} H_1^T H_2 + H_2^T H_1 & -H_1^T - H_2^T \\ -H_1 - H_2 & 2I\end{bmatrix}\begin{bmatrix} x_{kl}(t) \\ h_{kl}(t)\end{bmatrix} \leqslant 0. \tag{46}$$

Noting that $\mathcal{U}G_i^{(q)} = G_i^{(q)}\mathcal{U} = NG_i^{(q)}$ $(q = 1, 2)$, based on the properties of the Kronecker product(lemma 1), for any matrix H with an appropriate dimension, we obtain

$$(\mathcal{U}\otimes H)\big(G_i^{(q)}\otimes\Gamma_{qi}\big) = \big(\mathcal{U}G_i^{(q)}\big)\otimes(H\Gamma_{qi}) = \big(NG_i^{(q)}\big)\otimes(H\Gamma_{qi}). \tag{47}$$

Adding (28)–(37) to the right of (27) and substituting (38)–(46) into (27), we obtain

$$\begin{aligned}\mathcal{L}V(t, x_t, i) \leqslant \sum_{1\leqslant k<l\leqslant N} &\{2x_{kl}^T(t)P_i y_{kl}(t) + x_{kl}^T(t)\sum_{j=1}^m \gamma_{ij}P_j x_{kl}(t) - Nx_{kl}^T(t)G_{kli}^{(1,1)}\Gamma_{li}^T P_i\Gamma_{li}x_{kl}(t)\\ &- 2N\beta_0 x_{kl}^T(t-\tau_1(t))G_{kli}^{(2,2)}\Gamma_{2i}^T P_i\Gamma_{2i}x_{kl}(t-\tau_1(t))\\ &- 2N(1-\beta_0)x_{kl}^T(t-\tau_2(t))G_{kli}^{(2,2)}\Gamma_{2i}^T P_i\Gamma_{2i}x_{kl}(t-\tau_2(t))\\ &+ 2\beta_0\lambda_i x_{kl}^T(t)\Pi_1^T\Pi_1 x_{kl}(t) + 2\beta_0\lambda_i x_{kl}^T(t-\tau_1(t))\Pi_2^T\Pi_2 x_{kl}(t-\tau_1(t))\\ &+ 2(1-\beta_0)\lambda_i x_{kl}^T(t)\Pi_3^T\Pi_3 x_{kl}(t) + 2(1-\beta_0)\lambda_i x_{kl}^T(t-\tau_2(t))\Pi_4^T\Pi_4 x_{kl}(t-\tau_2(t))\\ &+ x_{kl}^T(t)(Q_1+Q_2)x_{kl}(t) - x_{kl}^T(t-\tau_1)Q_1x_{kl}(t-\tau_1) - x_{kl}^T(t-\tau_2)Q_2x_{kl}(t-\tau_2)\\ &+ y_{kl}^T(t)[\tau_1R_1 + (\tau_2-\tau_1)R_2]y_{kl}(t) + \tau^2h_{kl}^T(t)R_3h_{kl}(t)\\ &- \left[\int_{t-\tau}^t h_{kl}(s)\,\mathrm{d}s\right]^T R_3\left[\int_{t-\tau}^t h_{kl}(s)\,\mathrm{d}s\right]\\ &+ \beta_0(1-\beta_0)(\mathcal{B}\xi)_{kl}^T(t)[\tau_1Z_1 + (\tau_2-\tau_1)Z_2]\mathcal{B}\xi)_{kl}(t) + 2\xi_{kl}^T(t)Y[x_{kl}(t) - x_{kl}(t-\tau_1(t))]\\ &+ 2\xi_{kl}^T(t)M[x_{kl}(t-\tau_1(t)) - x_{kl}(t-\tau_1)] + 2\xi_{kl}^T(t)T[x_{kl}(t-\tau_1) - x_{kl}(t-\tau_2(t))]\\ &+ 2\xi_{kl}^T(t)W[x_{kl}(t-\tau_2(t)) - x_{kl}(t-\tau_2)] + 2\xi_{kl}^T(t)S[(\mathcal{A}\xi)_{kl}(t) - y_{kl}(t)]\\ &- \alpha_1\begin{bmatrix} x_{kl}(t) \\ f_{kl}(t)\end{bmatrix}^T \begin{bmatrix} F_1^T F_2 + F_2^T F_1 & -F_1^T - F_2^T \\ -F_1 - F_2 & 2I\end{bmatrix}\begin{bmatrix} x_{kl}(t) \\ f_{kl}(t)\end{bmatrix}\\ &- \alpha_2\begin{bmatrix} x_{kl}(t) \\ g_{kl}(t)\end{bmatrix}^T \begin{bmatrix} X_1^T X_2 + X_2^T X_1 & -X_1^T - X_2^T \\ -X_1 - X_2 & 2I\end{bmatrix}\begin{bmatrix} x_{kl}(t) \\ g_{kl}(t)\end{bmatrix}\\ &- \alpha_3\begin{bmatrix} x_{kl}(t) \\ h_{kl}(t)\end{bmatrix}^T \begin{bmatrix} H_1^T H_2 + H_2^T H_1 & -H_1^T - H_2^T \\ -H_1 - H_2 & 2I\end{bmatrix}\begin{bmatrix} x_{kl}(t) \\ h_{kl}(t)\end{bmatrix}\\ &+ \tau_1(t)\xi_{kl}^T(t)Y\big(R_1^{-1} + Z_1^{-1}\big)Y^T\xi_{kl}(t) + (\tau_1 - \tau_1(t))\xi_{kl}^T(t)M\big(R_1^{-1} + Z_1^{-1}\big)M^T\xi_{kl}(t)\\ &+ (\tau_2(t) - \tau_1)\xi_{kl}^T(t)T\big(R_2^{-1} + Z_2^{-1}\big)T^T\xi_{kl}(t) + (\tau_2 - \tau_2(t))\xi_{kl}^T(t)W\big(R_2^{-1} + Z_2^{-1}\big)W^T\xi_{kl}(t)\}\end{aligned}$$

$$= \sum_{1\leqslant k<l\leqslant N} \xi_{kl}^T(t)\{\Sigma_{11i} + \tau_1(t)Y\left(R_1^{-1} + Z_1^{-1}\right)Y^T + (\tau_1 - \tau_1(t))M\left(R_1^{-1} + Z_1^{-1}\right)M^T$$
$$+ (\tau_2(t) - \tau_1)T\left(R_2^{-1} + Z_2^{-1}\right)T^T + (\tau_2 - \tau_2(t))W\left(R_2^{-1} + Z_2^{-1}\right)W^T\}\xi_{kl}(t)$$
$$= \sum_{1\leqslant k<l\leqslant N} \xi_{kl}^T(t)\Xi\xi_{kl}(t). \tag{48}$$

By Schur complement, we can conclude from (23)

$$\Sigma_{11i} + \tau_1 M\left(R_1^{-1} + Z_1^{-1}\right)M^T + (\tau_2 - \tau_1)W\left(R_2^{-1} + Z_2^{-1}\right)W^T < 0$$
$$\Sigma_{11i} + \tau_1 M\left(R_1^{-1} + Z_1^{-1}\right)M^T + (\tau_2 - \tau_1)T\left(R_2^{-1} + Z_2^{-1}\right)T^T < 0$$
$$\Sigma_{11i} + \tau_1 Y\left(R_1^{-1} + Z_1^{-1}\right)Y^T + (\tau_2 - \tau_1)W\left(R_2^{-1} + Z_2^{-1}\right)W^T < 0$$
$$\Sigma_{11i} + \tau_1 Y\left(R_1^{-1} + Z_1^{-1}\right)Y^T + (\tau_2 - \tau_1)T\left(R_2^{-1} + Z_2^{-1}\right)T^T < 0.$$

By using lemma 4, we can obtain $\Xi < 0$, and then it follows that

$$\mathcal{L}V(t, x(t), i) \leqslant \lambda_{\max}(\Xi) \sum_{1\leqslant k<l\leqslant N} E\{\|\xi_{kl}(t)\|^2\}$$
$$\leqslant \lambda_{\max}(\Xi) \sum_{1\leqslant k<l\leqslant N} E\{\|x_{kl}(t)\|^2\}. \tag{49}$$

Therefore, we have

$$E\{V(t, x(t), i)\} - E\{V(0)\} \leqslant \lambda_{\max}(\Xi) \int_0^t \sum_{1\leqslant k<l\leqslant N} E\{\|x_{kl}(s)\|^2\}\, \mathrm{d}s, \tag{50}$$

which implies that

$$\int_0^t \sum_{1\leqslant k<l\leqslant N} E\{\|x_{kj}(s)\|^2\}\, \mathrm{d}s \leqslant -\frac{1}{\lambda_{\max}(\Xi)} E\{V(0)\} < +\infty. \tag{51}$$

Moreover, by lemma 5, it is not difficult to see that $E\{\|x_{kl}(s)\|^2\}$ is uniformly continuous on $[0, +\infty)$, that is

$$E\{\|x_{kl}(t)\|^2\} \to 0 \qquad t \to +\infty. \tag{52}$$

The proof is complete. □

Remark 4. From theorem 1, it can be seen that the feasibility of (23) and (24) depends on not only τ_1 and τ_2, but also the probability distribution of the delay-taking values in the interval; more information of the time delays is involved (23). Moreover, by the introduction of lemmas 3 and 4, the convexity of the matrix equations is employed to derive the criteria. Therefore, they may lead to a larger allowable upper bound of the time delays.

Model (9) is quite general. Let us now consider three special cases, and the corresponding results are still believed to be new.

Case 1. We first specialize system (9) to the case without external disturbance; then system (9) degenerates as follows:

$$\dot{x}_k(t) = A(r(t))f(x_k(t)) + B(r(t))g(x_k(t - \tau(t))) + C(r(t)) \int_{t-\tau}^t h(x_k(s))\, \mathrm{d}s$$
$$+ \sum_{l=1}^N G_{kl,r(t)}^{(1)} \Gamma_{1,r(t)} x_l(t) + \sum_{l=1}^N G_{kl,r(t)}^{(2)} \Gamma_{2,r(t)} x_l(t - \tau(t)) \quad (k = 1, 2, \ldots, N). \tag{53}$$

Similar to theorem 1, we can derive synchronization stability criterion for system (53).

Corollary 1. *For given scalars $\tau_2 > \tau_1 > 0, \tau > 0$ and $\beta_0 > 0$, the stochastic delayed networks (53) with mixed time delays and Markov switching are asymptotically synchronized in the mean square, if there exist appropriate dimensional matrices $P_i > 0\,(i = 1, 2, \ldots, m)$, $Q_i > 0$, $Z_i > 0\,(i = 1, 2)$, $R_j > 0\,(j = 1, 2, 3)$ and $Y_i, M_i, T_i, W_i\ (i = 1, 2)$ and $S_j\ (j = 1, 2, 3, 4)$ and positive scalars $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$ such that the following linear matrix inequalities hold for all $1 \leqslant k < l \leqslant N$:*

$$\begin{bmatrix} \bar{\Sigma}_{11i} & * \\ \Sigma_{21}^{l} & \Sigma_{22} \end{bmatrix} < 0 \qquad (i = 1, 2, \ldots, m; l = 1, 2, 3, 4) \tag{54}$$

where

$$\bar{\Sigma}_{11i} = \begin{bmatrix} \bar{\Theta}_{11i} & * & * & * & * & * & * & * & * & * & * \\ \Theta_{21i} & \bar{\Theta}_{22i} & * & * & * & * & * & * & * & * & * \\ 0 & \Theta_{32i} & \Theta_{33i} & * & * & * & * & * & * & * & * \\ \Theta_{41i} & \Theta_{42i} & \Theta_{43i} & \bar{\Theta}_{44i} & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \Theta_{54i} & \Theta_{55i} & * & * & * & * & * & * \\ \Theta_{61i} & A_i^T S_2^T & 0 & A_i^T S_3^T & 0 & -2\alpha_1 I & * & * & * & * & * \\ \Theta_{71i} & \Theta_{72i} & 0 & \Theta_{74i} & 0 & 0 & \Theta_{77i} & * & * & * & * \\ \Theta_{81i} & \Theta_{82i} & 0 & \Theta_{84i} & 0 & 0 & \Theta_{87i} & \Theta_{88i} & * & * & * \\ \Theta_{91i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Theta_{99i} & * & * \\ C_i^T S_1^T & C_i^T S_2^T & 0 & C_i^T S_3^T & 0 & 0 & 0 & 0 & 0 & -R_3 & * \\ \Theta_{11,1i} & \Theta_{11,2i} & 0 & \Theta_{11,4i} & 0 & S_4 A_i & \beta_0 S_4 B_i & \Theta_{11,8i} & 0 & S_4 C_i & \Theta_{11,11i} \end{bmatrix}$$

$$\begin{aligned} \bar{\Theta}_{11i} &= \sum_{j=1}^{m} \gamma_{ij} P_j + Q_1 + Q_2 + Y_1 + Y_1^T - N G_{kli}^{(1)} S_1 \Gamma_{1i} - N G_{kli}^{(1)} \Gamma_{1i}^T S_1^T - \alpha_1 \big(F_1^T F_2 + F_2^T F_1\big) \\ &\quad - 2\alpha_2 \big(X_1^T X_2 + X_2^T X_1\big) - \alpha_3 \big(H_1^T H_2 + H_2^T H_1\big) \\ \bar{\Theta}_{22i} &= -\beta_0 (1-\beta_0) N \tau_1 G_{kli}^{(2,2)} \Gamma_{2i}^T Z_1 \Gamma_{2i} - \beta_0 (1-\beta_0) N (\tau_2 - \tau_1) G_{kli}^{(2,2)} \Gamma_{2i}^T Z_2 \Gamma_{2i} \\ &\quad - Y_2 - Y_2^T + M_1 + M_1^T - \beta_0 N G_{kli}^{(2)} S_2 \Gamma_{2i} - \beta_0 N G_{kli}^{(2)} \Gamma_{2i}^T S_2^T \\ \bar{\Theta}_{44i} &= -\beta_0 (1-\beta_0) N \tau_1 G_{kli}^{(2,2)} \Gamma_{2i}^T Z_1 \Gamma_{2i} - \beta_0 (1-\beta_0) N (\tau_2 - \tau_1) G_{kli}^{(2,2)} \Gamma_{2i}^T Z_2 \Gamma_{2i} - T_2 - T_2^T \\ &\quad + W_1 + W_1^T - (1-\beta_0) N G_{kli}^{(2)} S_3 \Gamma_{2i} - (1-\beta_0) N G_{kli}^{(2)} \Gamma_{2i}^T S_3^T. \end{aligned}$$

Others terms are given in theorem 1, the proof is similar to that of theorem 1, which is omitted here.

Case 2. Let us assume that system (53) evolves with neither mode switching nor distributed time delay; then system (53) reduces to

$$\begin{aligned} \dot{x}_k(t) &= A f(x_k(t)) + B g(x_k(t-\tau(t))) + \sum_{l=1}^{N} G_{kl}^{(1)} \Gamma_1 x_l(t) \\ &\quad + \sum_{l=1}^{N} G_{kl}^{(2)} \Gamma_2 x_l(t-\tau(t)) \quad (k = 1, 2, \ldots, N). \end{aligned} \tag{55}$$

Similar to corollary 1, we can derive synchronization stability criterion for system (55).

Corollary 2. *For given scalars $\tau_2 > \tau_1 > 0$ and $\beta_0 > 0$, the stochastic delayed networks (55) are asymptotically synchronized in the mean square, if there exist appropriate dimensional matrices $P > 0$, $Q_i > 0$, $Z_i > 0$, $R_i > 0$ and $Y_i, M_i, T_i, W_i\ (i = 1, 2)$ and $S_j\ (j = 1, 2, 3, 4)$*

and positive scalars $\alpha_1 > 0, \alpha_2 > 0$ such that the following linear matrix inequalities hold for all $1 \leqslant k < l \leqslant N$:

$$\begin{bmatrix} \hat{\Sigma}_{11} & * \\ \Sigma_{21}^{l} & \Sigma_{22} \end{bmatrix} < 0 \qquad (l = 1, 2, 3, 4) \tag{56}$$

where

$$\hat{\Sigma}_{11} = \begin{bmatrix} \hat{\Theta}_{11} & * & * & * & * & * & * & * & * \\ \hat{\Theta}_{21} & \hat{\Theta}_{22} & * & * & * & * & * & * & * \\ 0 & M_2 - M_1^T & \hat{\Theta}_{33} & * & * & * & * & * & * \\ \hat{\Theta}_{41} & \hat{\Theta}_{42} & T_2^T - T_1 & \hat{\Theta}_{44} & * & * & * & * & * \\ 0 & 0 & 0 & W_2 - W_1^T & -Q_2 - W_2 - W_2^T & * & * & * & * \\ \hat{\Theta}_{61} & A^T S_2^T & 0 & A^T S_3^T & 0 & -2\alpha_1 I & * & * & * \\ \hat{\Theta}_{71} & \hat{\Theta}_{72} & 0 & \hat{\Theta}_{74} & 0 & 0 & \hat{\Theta}_{77} & * & * \\ \hat{\Theta}_{81} & \hat{\Theta}_{82} & 0 & \hat{\Theta}_{84} & 0 & 0 & \hat{\Theta}_{87} & \hat{\Theta}_{88} & * \\ \hat{\Theta}_{91} & \hat{\Theta}_{92} & 0 & \hat{\Theta}_{94} & 0 & S_4 A & \beta_0 S_4 B & (1-\beta_0) S_4 B & \hat{\Theta}_{99} \end{bmatrix}$$

$$\hat{\Theta}_{11} = Q_1 + Q_2 + Y_1 + Y_1^T - NG_{kli}^{(1)} S_1 \Gamma_{1i} - NG_{kli}^{(1)} \Gamma_{1i}^T S_1^T - \alpha_1 \big(F_1^T F_2 + F_2^T F_1\big) - 2\alpha_2 \big(X_1^T X_2 + X_2^T X_1\big)$$

$$\hat{\Theta}_{21} = Y_2 - Y_1^T - \beta_0 N G_{kl}^{(2)} \Gamma_2^T S_1^T - NG_{kl}^{(1)} S_2 \Gamma_1$$

$$\hat{\Theta}_{22} = -\beta_0(1-\beta_0) N \tau_1 G_{kl}^{(2,2)} \Gamma_2^T Z_1 \Gamma_2 - \beta_0(1-\beta_0) N(\tau_2 - \tau_1) G_{kl}^{(2,2)} \Gamma_2^T Z_2 \Gamma_2 - Y_2 - Y_2^T + M_1 + M_1^T - \beta_0 N G_{kl}^{(2)} S_2 \Gamma_2 - \beta_0 N G_{kl}^{(2)} \Gamma_2^T S_2^T$$

$$\hat{\Theta}_{33} = -Q_1 - M_2 - M_2^T + T_1 + T_1^T$$

$$\hat{\Theta}_{41} = -(1-\beta_0) N G_{kl}^{(2)} \Gamma_2^T S_1^T - NG_{kl}^{(1)} S_3 \Gamma_1$$

$$\hat{\Theta}_{42} = \beta_0(1-\beta_0) N \tau_1 G_{kl}^{(2,2)} \Gamma_2^T Z_1 \Gamma_2 + \beta_0(1-\beta_0) N(\tau_2 - \tau_1) G_{kl}^{(2,2)} \Gamma_2^T Z_2 \Gamma_2 - (1-\beta_0) N G_{kl}^{(2)} \Gamma_2^T S_2^T - \beta_0 N G_{kl}^{(2)} S_3 \Gamma_2$$

$$\hat{\Theta}_{44} = -\beta_0(1-\beta_0) N \tau_1 G_{kl}^{(2,2)} \Gamma_2^T Z_1 \Gamma_2 - \beta_0(1-\beta_0) N(\tau_2 - \tau_1) G_{kl}^{(2,2)} \Gamma_2^T Z_2 \Gamma_2 - T_2 - T_2^T + W_1 + W_1^T - (1-\beta_0) N G_{kl}^{(2)} S_3 \Gamma_2 - (1-\beta_0) N G_{kl}^{(2)} \Gamma_2^T S_3^T$$

$$\hat{\Theta}_{61} = A^T S_1^T + \alpha_1 (F_1 + F_2)$$

$$\hat{\Theta}_{71} = \beta_0 B^T S_1^T + \alpha_2 (X_1 + X_2)$$

$$\hat{\Theta}_{72} = -\beta_0(1-\beta_0) N \tau_1 G_{kl}^{(2,2)} \Gamma_2^T Z_1 \Gamma_2 - \beta_0(1-\beta_0) N(\tau_2 - \tau_1) G_{kl}^{(2,2)} \Gamma_2^T Z_2 \Gamma_2 + \beta_0 B^T S_2^T$$

$$\hat{\Theta}_{74} = \beta_0(1-\beta_0) N \tau_1 G_{kl}^{(2,2)} \Gamma_2^T Z_1 \Gamma_2 + \beta_0(1-\beta_0) N(\tau_2 - \tau_1) G_{kl}^{(2,2)} \Gamma_2^T Z_2 \Gamma_2 + \beta_0 B^T S_3^T$$

$$\hat{\Theta}_{77} = \beta_0(1-\beta_0) \tau_1 B^T Z_1 B + \beta_0(1-\beta_0)(\tau_2 - \tau_1) B^T Z_2 B - 2\alpha_2 I$$

$$\hat{\Theta}_{81} = (1-\beta_0) B^T S_1^T + \alpha_2 (X_1 + X_2)$$

$$\hat{\Theta}_{82} = \beta_0(1-\beta_0) N \tau_1 B^T Z_1 \Gamma_2 + \beta_0(1-\beta_0) N(\tau_2 - \tau_1) B^T Z_2 \Gamma_2 + (1-\beta_0) B^T S_2^T$$

$$\hat{\Theta}_{84} = -\beta_0(1-\beta_0) N \tau_1 B^T Z_1 \Gamma_2 - \beta_0(1-\beta_0) N(\tau_2 - \tau_1) B^T Z_2 \Gamma_2 + (1-\beta_0) B^T S_3^T$$

$$\hat{\Theta}_{87} = -\beta_0(1-\beta_0) \tau_1 B^T Z_1 B - \beta_0(1-\beta_0)(\tau_2 - \tau_1) B^T Z_2 B$$

$$\hat{\Theta}_{88} = \beta_0(1-\beta_0) \tau_1 B^T Z_1 B + \beta_0(1-\beta_0)(\tau_2 - \tau_1) B^T Z_2 B - 2\alpha_2 I$$

$$\hat{\Theta}_{91} = P - NG_{kl}^{(1)} S_4 \Gamma_1 - S_1^T$$

$$\hat{\Theta}_{92} = -\beta_0 N G_{kl}^{(2)} S_4 \Gamma_2 - S_2^T$$

$\hat{\Theta}_{94} = -(1-\beta_0)NG_{kl}^{(2)}S_4\Gamma_2 - S_3^T$
$\hat{\Theta}_{98} = (1-\beta_0)S_4B$
$\hat{\Theta}_{99} = \tau_1 R_1 + (\tau_2-\tau_1)R_2 - S_4 - S_4^T.$

The proof can be obtained by following the same line in theorem 1.

Case 3. In this case, we consider the system (2) with $\beta(t) = 1$, that is, the discrete time delay $\tau(t) \in [0, \tau_2]$, the system (2) degenerates as follows:

$$\begin{aligned} dx(t) = {} & [(I_N\otimes A_i)F(x(t)) + (I_N\otimes B_i)G(x(t-\tau(t))) \\ & + (I_N\otimes C_i)\int_{t-\tau}^{t} H(x(s))\,ds]\,dt + \left(G_i^{(1)}\otimes\Gamma_{1i}\right)x(t) \\ & \times(dt + dw_1(t)) + \left(G_i^{(2)}\otimes\Gamma_{2i}\right)x(t-\tau(t))(dt + dw_2(t)) + \sigma(t)\,dw_3(t). \end{aligned} \tag{57}$$

Define

$$y(t) = \mathcal{A}\xi(t) \tag{58}$$

where

$$\begin{aligned} \mathcal{A} &= \left[G_i^{(1)}\otimes\Gamma_{1i} \quad G_i^{(2)}\otimes\Gamma_{2i} \quad 0 \quad I_N\otimes A_i \quad I_N\otimes B_i \quad I_N\otimes C_i \quad 0\right] \\ \xi^T(t) &= \left[x^T(t) \quad x^T(t-\tau(t)) \quad x^T(t-\tau_2) \quad F^T(x(t)) \quad G^T(x(t-\tau(t))) \right. \\ & \qquad \left. H^T(x(t)) \quad \int_{t-\tau}^{t} H^T(x(s))\,ds \quad y^T(t)\right]; \end{aligned}$$

then system (57) can be expressed as

$$dx(t) = y(t)\,dt + \bar{\sigma}(t)\,dw(t) \tag{59}$$

where

$$\begin{aligned} \bar{\sigma}(t) &= \left[\left(G_i^{(1)}\otimes\Gamma_{1i}\right)x(t) \quad \left(G_i^{(2)}\otimes\Gamma_{2i}\right)x(t-\tau(t)) \quad \sigma(t)\right] \\ dw^T(t) &= \left[dw_1^T(t) \quad dw_2^T(t) \quad dw_3^T(t)\right]. \end{aligned}$$

Corollary 3. *For given scalars* $\tau_2 > 0$ *and* $\tau > 0$*, the stochastic delayed networks (57) with mixed time delays and Markov switching are asymptotically synchronized in the mean square, if there exist appropriate dimensional matrices* $P_i > 0 (i = 1, 2, \ldots, m)$, $R_i > 0\,(i = 1, 2)$, $Q_1 > 0$ *and* $Y_i, M_i\ (i = 1, 2)$ *and* $S_j\ (j = 1, 2, 3)$ *and positive scalars* $\lambda_1 > 0, \lambda_2 > 0, \alpha_1 > 0, \alpha_2 > 0, \alpha_3 > 0$ *such that the following linear matrix inequalities hold for all* $1 \leqslant k < l \leqslant N$*:*

$$\begin{bmatrix} \tilde{\Sigma}_{11i} & * \\ \tilde{\Sigma}_{21}^{l} & \tilde{\Sigma}_{22} \end{bmatrix} < 0 \tag{60}$$

$$P_i - \lambda_i I < 0 \qquad (i = 1, 2, \ldots, m;\ l = 1, 2) \tag{61}$$

where

$$\tilde{\Sigma}_{11i} = \begin{bmatrix} \tilde{\Theta}_{11i} & * & * & * & * & * & * & * \\ \tilde{\Theta}_{21i} & \tilde{\Theta}_{22i} & * & * & * & * & * & * \\ 0 & \tilde{\Theta}_{32i} & \tilde{\Theta}_{33i} & * & * & * & * & * \\ \tilde{\Theta}_{41i} & A_i^T S_2^T & 0 & -2\alpha_1 & * & * & * & * \\ \tilde{\Theta}_{51i} & B_i^T S_2^T & 0 & 0 & -2\alpha_2 & * & * & * \\ \tilde{\Theta}_{61i} & 0 & 0 & 0 & 0 & \tilde{\Theta}_{66i} & * & * \\ C_i^T S_1^T & C_i^T S_2^T & 0 & 0 & 0 & 0 & -R_2 & * \\ \tilde{\Theta}_{81i} & \tilde{\Theta}_{82i} & 0 & S_3A_i & S_3B_i & 0 & S_3C_i & \Theta_{88i} \end{bmatrix}$$

$$\tilde{\Sigma}_{21}^{1} = \begin{bmatrix} \tau_2 Y_1^T & \tau_2 Y_2^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\tilde{\Sigma}_{21}^{2} = \begin{bmatrix} 0 & \tau_2 M_1^T & \tau_2 M_2^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\tilde{\Sigma}_{22} = -\tau_2 R_1$$
$$\tilde{\Theta}_{11i} = \sum_{j=1}^{m} \gamma_{ij} P_j - NG_{kli}^{(1,1)} \Gamma_{1i}^T P_i \Gamma_{1i} + \lambda_i \Pi_1^T \Pi_1 + Q_1 + Y_1 + Y_1^T - NG_{kli}^{(1)} S_1 \Gamma_{1i} - NG_{kli}^{(1)} \Gamma_{1i}^T S_1^T$$
$$- \alpha_1 \left(F_1^T F_2 + F_2^T F_1\right) - \alpha_2 \left(X_1^T X_2 + X_2^T X_1\right) - \alpha_3 \left(H_1^T H_2 + H_2^T H_1\right)$$
$$\tilde{\Theta}_{21i} = Y_2 - Y_1^T - NG_{kli}^{(2)} \Gamma_{2i}^T S_1^T - NG_{kli}^{(1)} S_2 \Gamma_{1i}$$
$$\tilde{\Theta}_{22i} = -2NG_{kli}^{(2,2)} \Gamma_{2i}^T P_i \Gamma_{2i} + \lambda_i \Pi_2^T \Pi_2 - Y_2 - Y_2^T + M_1 + M_1^T - NG_{kli}^{(2)} S_2 \Gamma_{2i} - NG_{kli}^{(2)} \Gamma_{2i}^T S_2^T$$
$$\tilde{\Theta}_{32i} = M_2 - M_1^T$$
$$\tilde{\Theta}_{33i} = -Q_1 - M_2 - M_2^T$$
$$\tilde{\Theta}_{41i} = A_i^T S_1^T + \alpha_1 (F_1 + F_2)$$
$$\tilde{\Theta}_{51i} = B_i^T S_1^T + \alpha_2 (X_1 + X_2)$$
$$\tilde{\Theta}_{61i} = \alpha_3 (H_1 + H_2)$$
$$\tilde{\Theta}_{66i} = \tau^2 R_2 - 2\alpha_3$$
$$\tilde{\Theta}_{81i} = P_i - NG_{kli}^{(1)} S_3 \Gamma_{1i} - S_1^T$$
$$\tilde{\Theta}_{82i} = -NG_{kli}^{(2)} S_3 \Gamma_{2i} - S_2^T$$
$$\tilde{\Theta}_{88i} = \tau_2 R_1 - S_3 - S_3^T.$$

Proof. Construct a Lyapunov–Krasovskii functional candidate as

$$V(t, x(t), i) = V_1(t, x(t), i) + V_2(t, x(t), i) + V_3(t, x(t), i) \tag{62}$$

where

$$V_1(t, x(t), i) = x^T(t)(\mathcal{U} \otimes P_i)x(t)$$
$$V_2(t, x(t), i) = \int_{t-\tau_2}^{t} x^T(s)(\mathcal{U} \otimes Q_1)x(s)\,\mathrm{d}s$$
$$V_3(t, x(t), i) = \int_{t-\tau_2}^{t} \int_{s}^{t} y^T(v)(\mathcal{U} \otimes R_1)y(v)\,\mathrm{d}v\,\mathrm{d}s + \tau \int_{t-\tau}^{t} \int_{s}^{t} H^T(x(v))(\mathcal{U} \otimes R_2)H(x(v))\,\mathrm{d}v\,\mathrm{d}s$$

where P_i $(i = 1, 2, \ldots, m)$, Q_1, R_1, $R_2 > 0$; the following parts can be obtained using the similar method in theorem 1. □

Remark 5. The complex network model proposed in this paper is fairly comprehensive, which comprises stochastic, Markovian jumping characteristics, discrete time delays and distributed time delays. Therefore, our main results can readily specialize to many special cases, such as stochastic complex networks with mixed time delays, stochastic complex networks with Markovian jumping parameters and stochastic complex networks with Markovian switching. The specialized results are still believed to be new as the network models have not been fully researched yet. For presentation, we omit the corresponding corollaries here.

4. A numerical example

In this section, we use a number of examples to illustrate the results derived in this work. The above synchronization conditions can be applied to networks with different topologies and different sizes. In order to illustrate the main results, we consider a lower dimensional network model.

Example 1. For simplicity, we consider complex dynamical networks with three nodes and the state vector of each node being two dimensional, i.e. $N = 3$, $n = 2$; other parameters are given as follows:

$$A_1 = \begin{bmatrix} -12 & 0 \\ 0 & -14 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0.8 & 0.8 \\ -1.2 & -1.6 \end{bmatrix} \qquad C_1 = \begin{bmatrix} -1.4 & -1 \\ 0.8 & 1.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -10 & 0 \\ 0 & -12 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1.2 & -1 \\ -0.8 & -0.6 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 1.4 & 1 \\ -0.6 & 0.6 \end{bmatrix}$$

$$G_1^{(1)} = \begin{bmatrix} -0.2 & 0.1 & 0.1 \\ 0.1 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.2 \end{bmatrix} \quad G_2^{(1)} = 2G_1^{(1)} \quad \Gamma_{11} = \begin{bmatrix} 0.7 & -0.6 \\ -0.4 & 0.7 \end{bmatrix} \quad \Gamma_{12} = 0.1\Gamma_{11}$$

$$G_1^{(2)} = \begin{bmatrix} -0.3 & 0.1 & 0.2 \\ 0.1 & -0.2 & 0.1 \\ 0.2 & 0.1 & -0.3 \end{bmatrix} \quad G_2^{(2)} = 2G_1^{(2)} \quad \Gamma_{21} = \begin{bmatrix} -0.3 & 0.25 \\ -0.35 & -0.4 \end{bmatrix} \quad \Gamma_{22} = 0.2\Gamma_{21}$$

$$\Pi_1 = \Pi_2 = \Pi_3 = \Pi_4 = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -0.1 \end{bmatrix} \quad \gamma_{11} = -3 \quad \gamma_{12} = 3 \quad \gamma_{21} = 4 \quad \gamma_{22} = -4.$$

Furthermore, we take the following nonlinear functions:

$$f(x_k(t)) = [0.5x_{k1}(t) - \tanh(0.2x_{k1}(t)) + 0.2x_{k2}(t) \quad 0.95x_{k2}(t) - \tanh(0.75x_{k2}(t))\,]^T$$
$$g(x_k(t)) = [0.2x_{k1}(t) - \tanh(0.1x_{k1}(t)) \quad 0.1x_{k2}(t)\,]^T$$
$$h(x_k(t)) = [0.2x_{k1}(t) - \tanh(0.1x_{k1}(t)) \quad 0.1x_{k2}(t)\,]^T.$$

Then, it is easy to verify that

$$F_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.2 \end{bmatrix} \qquad F_2 = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.95 \end{bmatrix} \qquad X_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$X_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad H_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

When the delay is random and its probability distribution is known *a priori*, let $\tau = 1, \tau_1 = 0.5$ and $\beta_0 = 0.9$; by using theorem 1, it is found that the maximum delay bound is $\tau_2 = 4.619$, for which the synchronized states of the networks are asymptotically stable. For example, in theorem 1, when $\tau_2 = 2$, by employing the LMI toolbox in MATLAB, we can find a feasible solution with the following matrix variables (for simplicity, only the positive matrix variables used in theorem 1 are listed):

$$P_1 = \begin{bmatrix} 0.0302 & 0.0078 \\ 0.0078 & 0.0519 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0.0321 & 0.0069 \\ 0.0069 & 0.0549 \end{bmatrix} \qquad Q_1 = \begin{bmatrix} 0.0361 & 0.0206 \\ 0.0206 & 0.0603 \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} 0.0367 & 0.0212 \\ 0.0212 & 0.0595 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0.0051 & 0.0004 \\ 0.0004 & 0.0043 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 0.0019 & 0.0003 \\ 0.0003 & 0.0017 \end{bmatrix}$$
$$R_3 = \begin{bmatrix} 0.2227 & 0.0439 \\ 0.0439 & 0.2509 \end{bmatrix} \qquad Z_1 = \begin{bmatrix} 0.3085 & 0.0097 \\ 0.0097 & 0.2749 \end{bmatrix} \qquad Z_2 = \begin{bmatrix} 0.1105 & 0.0043 \\ 0.0043 & 0.0981 \end{bmatrix}$$
$$\lambda_1 = 0.0651 \qquad \lambda_2 = 0.0748 \qquad \alpha_1 = 0.6205 \qquad \alpha_2 = 0.1164 \qquad \alpha_3 = 0.2288.$$

To provide relatively complete information, we can obtain table 1 by using the LMI toolbox in MATLAB, which lists the maximum allowable bounds for different τ_1 and β_0. It can be found from table 1 that, when the probability distribution of the time delay can be observed,

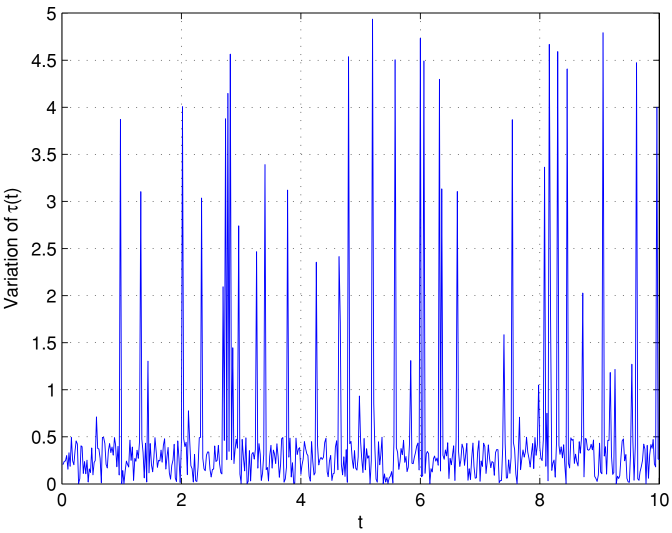

Figure 1. Variation of $\tau(t)$ with $\beta_0 = 0.9$, $\tau_1 = 0.5$ and $\tau_2 = 5$.

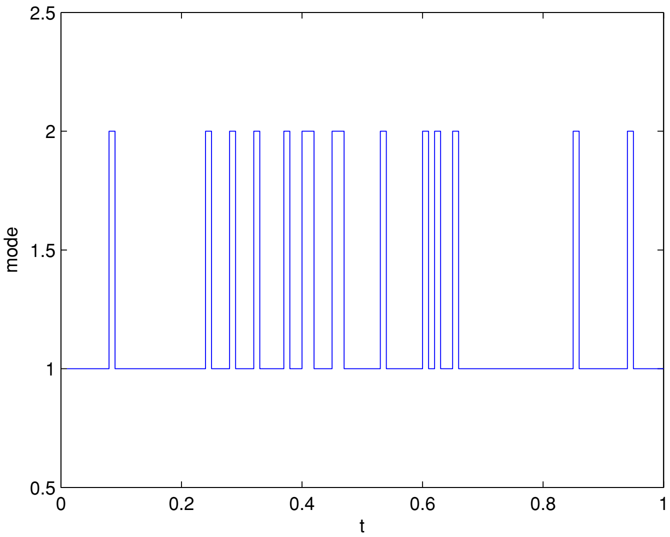

Figure 2. The curve of the operation modes of example 1.

using theorem 1 can lead to a larger allowable upper bound of the delay than that using only the variation range of the delay (the allowable upper bound of τ_2 is 2.047 by corollary 3). Figures 1 and 2 show variation of $\tau(t)$ with $\beta_0 = 0.9$, $\tau_1 = 0.5$, $\tau_2 = 5$ and the curve of the operation modes, respectively.

We define the synchronization error of delayed complex networks as follows:

$$e(t) = \sum_{1\leqslant k<l\leqslant 3} \sum_{i=1}^{2} (x_{ki}(t) - x_{li}(t))^2.$$

Figure 3 depicts the curve of the synchronization error for randomly chosen initial conditions for the case with $\beta_0 = 0.9$, $\tau = 1$,$\tau_1 = 0.5$ and $\tau_2 = 4.5$; this figure shows the error converge

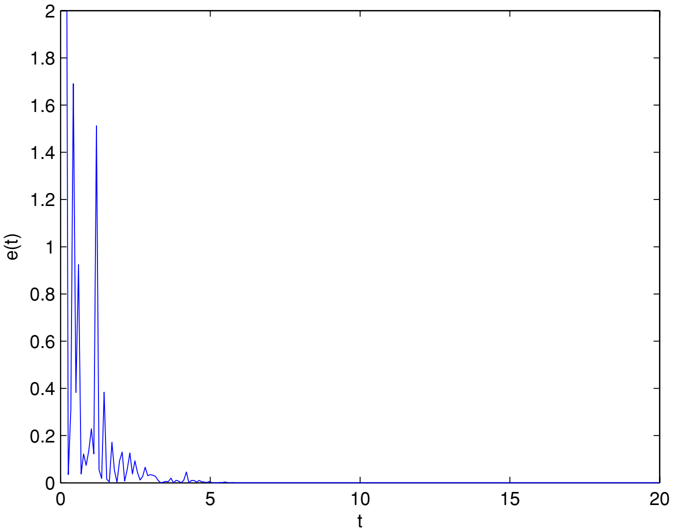

Figure 3. The curve of the synchronization error with $\beta_0 = 0.9$, $\tau_1 = 0.5$, $\tau_2 = 4.5$ and $\tau = 1$.

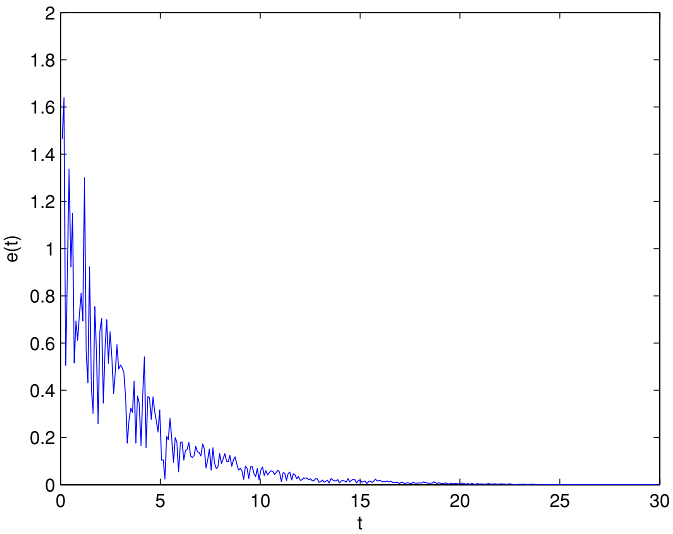

Figure 4. The curve of the synchronization error with $\beta_0 = 0.9$, $\tau_1 = 0.5$, $\tau_2 = 5$ and $\tau = 1$.

Table 1. Allowable upper bound of τ_2 for different β_0 and τ_1 ($\tau = 1$).

β_0	0.4	0.5	0.6	0.7	0.8	0.9	0.99
$\tau_1 = 0.5$	2.190	2.341	2.551	2.865	2.399	4.619	13.729
$\tau_1 = 1$	2.524	2.610	2.734	2.926	3.268	4.073	10.419

to zero under the above conditions. It can be seen from table 1 that, when $\beta_0 = 0.9$, $\tau = 1$, $\tau_1 = 0.5$, the allowable upper bound of τ_2 is 4.541. Since the LMI conditions are sufficient and not necessary, here we let $\tau_2 = 5$ (slighting above the threshold $\tau_2 = 4.619$). Figure 4

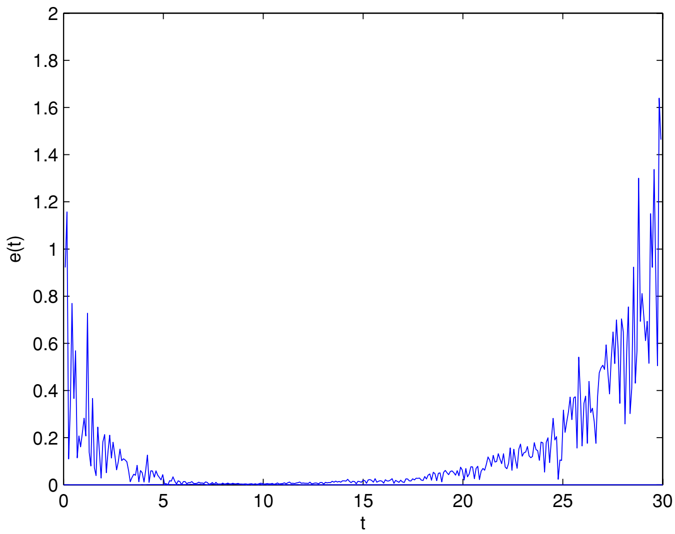

Figure 5. The curve of the synchronization error with $\beta_0 = 0.9$, $\tau_1 = 0.5$, $\tau_2 = 5.5$ and $\tau = 1$.

depicts the curve of the synchronization for randomly chosen initial conditions; this figure shows that the error converges to zero. But when we choose $\tau_2 = 5.5$ (sufficiently above threshold $\tau_2 = 4.619$), figure 5 depicts the curve of the synchronization; this figure shows that the error does not converge to zero under the above conditions. It is important to note that the obtained maximum delay bound ($\tau_2 = 4.619$) is close to the true value of the maximum delay bound beyond which the synchronized states are not asymptotically stable.

5. Conclusions

In this paper, we have dealt with the synchronization stability problem for a class of stochastic Markovian complex dynamical networks with distributed time delays and probabilistic interval discrete time delays. Based on the information of the probability distribution of time delay, some new models of the systems, which have stochastic parameter matrices, have been proposed. Based on the stochastic analysis techniques and the properties of the Kronecker product, some sufficient conditions for delay-dependent synchronization stability in the mean square are derived in the form of linear matrix inequalities. An example is presented to show the efficiency of the derived results. It should be pointed out that the method in the present paper can also be extended to the case when the probability of the delay-taking values in series of intervals can be observed; this work will be left for our future research.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (nos 60474079, 60774060, 60834002).

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